ALBANIAN JOURNAL OF MATHEMATICS
Volume 14, Number 1, Pages 03–23
ISSN: 1930-1235; (**2020**)

ON GROMOV-WITTEN THEORY OF TORIC GERBES

HSIAN-HUA TSENG

Department of Mathematics
Ohio State University
100 Math Tower, 231 West 18th Ave.
Columbus, OH 43210. USA

Abstract. Toric gerbes are étale gerbes over toric Deligne-Mumford stacks which are constructed out of suitably chosen toric data. In this paper we study the genus 0 Gromov-Witten theory of toric gerbes. Our main result equates the genus 0 Gromov-Witten theory of a toric gerbe with a suitable twist of the genus 0 Gromov-Witten theory of a disjoint union of several copies of the base. Our result can be interpreted in the context of the decomposition conjecture in physics. The main tool used in this paper is the calculation of Gromov-Witten theory of toric Deligne-Mumford stacks by Coates-Corti-Iritani-Tseng.

MSC 2010: 14N35, 14M25, 14A20
Keywords: Toric gerbes, Gromov-Witten theory.

1. Introduction

1.1. **Weighted projective stacks.** Given natural numbers $a_0, ..., a_n$, consider the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ given by

$$\lambda \cdot (z_0, ..., z_n) := (\lambda^{a_0} z_0, ..., \lambda^{a_n} z_n), \quad \lambda \in \mathbb{C}^*, (z_0, ..., z_n) \in \mathbb{C}^{n+1}.$$

The scheme-theoretic quotient

$$P(a_0, ..., a_n) := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$$

is called a weighted projective space and is a projective variety with at worst quotient singularities. It is a natural generalization of projective spaces and provides an interesting class of *toric varieties*.

For $r > 0$, it is a basic fact that

$$P(ra_0, ..., ra_n) \simeq P(a_0, ..., a_n)$$

E-mail address: hhtseng@math.ohio-state.edu.

as varieties. More precisely, this isomorphism is given by the map $[z_0, ..., z_n] \mapsto [z_0^r, ..., z_n^r]$ in homogeneous coordinates.

The fact that the quotient definition of weighted projective spaces resulted in singular varieties naturally leads to the consideration of quotients as *stacks*. The stack quotient

$$\mathbb{P}(a_0, ..., a_n) := [(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*],$$

called a weighted projective stack, is a nonsingular projective stack.

The isomorphism $P(ra_0, ..., ra_n) \to P(a_0, ..., a_n)$ lifts to a morphism of stacks

$$\mathbb{P}(ra_0, ..., ra_n) \to \mathbb{P}(a_0, ..., a_n)$$

which is *not* an isomorphism of stacks. This is an example of μ_r-gerbes, which in general means the following. Let G be a finite group and $\mathfrak{X}$ be a Deligne-Mumford stack. A G-gerbe over $\mathfrak{X}$ can be understood as a principal BG-bundle $\mathfrak{Y} \to \mathfrak{X}$ over $\mathfrak{X}$. Here $BG \simeq [\mathrm{pt}/G]$ is the classifying stack of the finite group G. Gerbes over Deligne-Mumford stacks for a class of Deligne-Mumford stacks which are expected to have nice geometric properties.

Since weighted projective spaces are examples of toric varieties, we view and study the μ_r-gerbe $\mathbb{P}(ra_0, ..., ra_n) \to \mathbb{P}(a_0, ..., a_n)$ as *toric gerbes*.

1.2. **Toric gerbes.** In this paper, we study Gromov-Witten theory of toric gerbes. The notion of toric Deligne-Mumford stacks is introduced in [6] as stacks $\mathfrak{X}(\mathbf{\Sigma})$ naturally associated to certain combinatorial data called the *stacky fans* $\mathbf{\Sigma} = (N, \Sigma, \beta)$. This notion is reviewed in Section 2.1. A toric gerbe is a morphism

$$\mathfrak{X}(\mathbf{\Sigma}) \to \mathfrak{X}(\mathbf{\Sigma}')$$

defined by a specific kind of morphism $\mathbf{\Sigma} \to \mathbf{\Sigma}'$ of stacky fans. Details are spelled out in Section 2.1. According to [14] and [20], a toric gerbe $\mathfrak{X}(\mathbf{\Sigma}) \to \mathfrak{X}(\mathbf{\Sigma}')$ can be obtained as an iterated root gerbe over $\mathfrak{X}(\mathbf{\Sigma}')$. In particular a toric gerbe is an A-gerbe, where A is a finite abelian group.

Our study of Gromov-Witten theory of étale gerbes is inspired by the physics paper [18]. A physics conjecture proposed in *loc. cit.*, when applied to the toric gerbe case, states that conformal field theories on $\mathfrak{X}(\mathbf{\Sigma})$ are equivalent to conformal field theories on a disjoint union of several copies of $\mathfrak{X}(\mathbf{\Sigma}')$ twisted by certain B-field. We will often call this the *decomposition conjecture*[1]. A more detailed discussion on decomposition conjecture in mathematics can be found in [22].

The main results of this paper can be understood as establishing versions of this decomposition conjecture in various contexts. At the level of Chen-Ruan orbifold cohomology, we show that the Chen-Ruan orbifold cohomology ring of $\mathfrak{X}(\mathbf{\Sigma})$ is indeed isomorphic to a direct sum of copies of the Chen-Ruan orbifold cohomology ring of $\mathfrak{X}(\mathbf{\Sigma}')$. This is done by explicit computations using the results of [6], see Theorem 3.3. At the level of Gromov-Witten theory we prove results equating the genus 0 Gromov-Witten theory of $\mathfrak{X}(\mathbf{\Sigma})$ with the genus 0 Gromov-Witten theory of a disjoint union of several copies of $\mathfrak{X}(\mathbf{\Sigma}')$ with suitable twists. This is done by applying some sophisticated techniques in toric Gromov-Witten theory, including the calculations of genus 0 Gromov-Witten theory of toric stacks [10]. We also point out that reconstruction techniques can be used to deduces results on higher genus Gromov-Witten theory. Details are presented in Section 4.

[1]It is also called *gerbe duality conjecture*.

When the base is a $\mathbb{P}^1$-stack with at most two cyclic stack points, our results have also been proven by P. Johnson [21] by a completely different method. The decomposition conjecture for trivial gerbes over an arbitrary base is proven in [3]. It is also proven in genus 0 Gromov-Witten theory for root gerbes over smooth projective varieties in [4]. Gromov-Witten theory of G-gerbes over 0-dimensional bases of the form BQ, with Q a finite group, is studied in [22]. The decomposition conjecture for Gromov-Witten theory of G-gerbes with *trivial bands*, which include toric gerbes, has been proven in [5] and [23]. Our method for studying toric gerbes is different and more combinatorial.

The bulk of this paper is organized as follows. We review basic properties of toric gerbes and Gromov-Witten theory in Section 2. We prove the decomposition of Chen-Ruan orbifold cohomology ring in Section 3. The decomposition of full genus 0 Gromov-Witten theory of toric gerbes is studied in Section 4. As an example, we work out in Example 4.5 some details of the case of μ_2-gerbe[2] $\mathbb{P}(4,6) \to \mathbb{P}(2,3)$.

1.3. **Acknowledgments.** The author thank D. Abramovich, A. Bayer, K. Behrend, B. Fantechi, P. Johnson, A. Kresch, Y. Ruan and A. Vistoli for valuable discussions. The author is also grateful to E. Andreini, T. Coates, A. Corti, H. Iritani, Y. Jiang, and X. Tang for related collaborations.

2. Preliminaries

In this Section we discuss some background materials on toric stacks and Gromov-Witten theory, which are used throughout this paper.

2.1. **Basics on toric gerbes.** Following [6], a *toric Deligne-Mumford stack* is defined in terms of a stacky fan

$$\mathbf{\Sigma} = (N, \Sigma, \beta),$$

where N is a finitely generated abelian group, $\Sigma \subset N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ is a simplicial fan and $\beta : \mathbb{Z}^n \to N$ is a map determined by the elements $\{b_1, \cdots, b_n\}$ in N. By assumption, β has finite cokernel and the images of b_i's under the natural map $N \to N_{\mathbb{Q}}$ generate the simplicial fan Σ. The toric Deligne-Mumford stack $\mathfrak{X}(\mathbf{\Sigma})$ associated to $\mathbf{\Sigma}$ is defined to be the quotient stack

$$\mathfrak{X}(\mathbf{\Sigma}) := [Z/G],$$

where Z is the open subvariety $\mathbb{C}^n \setminus \mathbb{V}(J_\Sigma)$, J_Σ is the irrelevant ideal of the fan, and G is the product of an algebraic torus and a finite abelian group. The G-action on Z is given by a group homomorphism $\alpha : G \to (\mathbb{C}^*)^n$, where α is obtained by applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the Gale dual $\beta^\vee : \mathbb{Z}^n \to N^\vee$ of β and $G = \mathrm{Hom}_{\mathbb{Z}}(N^\vee, \mathbb{C}^*)$.

Every stacky fan $\mathbf{\Sigma}$ has an underlying *reduced* stacky fan $\mathbf{\Sigma_{red}} = (\overline{N}, \Sigma, \overline{\beta})$, where $\overline{N} := N/N_{tor}$, $\overline{\beta} : \mathbb{Z}^n \to \overline{N}$ is the natural projection given by the vectors $\{\overline{b}_1, \cdots, \overline{b}_n\} \subseteq \overline{N}$. With these data one gets a toric Deligne-Mumford stack $\mathfrak{X}(\mathbf{\Sigma_{red}}) = [Z/\overline{G}]$, where $\overline{G} = \mathrm{Hom}_{\mathbb{Z}}(\overline{N}^\vee, \mathbb{C}^*)$ and $\overline{N}^\vee$ is the Gale dual $\overline{\beta}^\vee : \mathbb{Z}^n \to \overline{N}^\vee$ of the map $\overline{\beta}$. The stack $\mathfrak{X}(\mathbf{\Sigma_{red}})$ is a toric orbifold[3], and can be obtained by rigidifying $\mathfrak{X}(\mathbf{\Sigma})$.

Throughout this paper, we assume that $\mathfrak{X}(\mathbf{\Sigma})$ and $\mathfrak{X}(\mathbf{\Sigma_{red}})$ are projective stacks.

[2] A reason to be interested in $\mathbb{P}(4,6)$ is that it is isomorphic to the moduli stack $\overline{\mathcal{M}}_{1,1}$ of 1-pointed stable curves of arithmetic genus 1.

[3] I.e. the generic stabilizer is trivial.

Given a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$, one can consider the set Box defined as follows. For a cone $\sigma \in \Sigma$, define

$$\mathrm{Box}(\sigma) := \left\{ b \in N | \bar{b} = \sum_{\bar{b}_i \in \sigma} a_i \bar{b}_i, 0 \leq a_i < 1 \right\},$$

and set $\mathrm{Box}(\mathbf{\Sigma}) := \bigcup_{\sigma \in \Sigma} \mathrm{Box}(\sigma)$. We will also need the *closed box* of $\mathbf{\Sigma}$, defined as follows. For a cone $\sigma \in \Sigma$, define

$$\overline{\mathrm{Box}}(\sigma) := \left\{ b \in N | \bar{b} = \sum_{\bar{b}_i \in \sigma} a_i \bar{b}_i, 0 \leq a_i \leq 1 \right\},$$

and set $\overline{\mathrm{Box}}(\mathbf{\Sigma}) := \bigcup_{\sigma \in \Sigma} \overline{\mathrm{Box}}(\sigma)$.

We now come to the notion of toric gerbes. Suppose that there is a splitting of abelian groups

$$N = N' \oplus A,$$

where A is a finite abelian group. We define another stacky fan $\mathbf{\Sigma}' := (N', \Sigma', \beta')$ where $\Sigma' = \Sigma \subset N_{\mathbb{Q}} = N'_{\mathbb{Q}}$ and $\beta' : \mathbb{Z}^n \to N'$ is given by the vectors $\{b'_1, ..., b'_n\} \subset N'$ which are images of $b_1, ..., b_n$ under the natural projection $N \to N'$. By construction this yields a map $\mathbf{\Sigma} \to \mathbf{\Sigma}'$, which in turn induces a map

$$\mathfrak{X}(\mathbf{\Sigma}) \to \mathfrak{X}(\mathbf{\Sigma}') \tag{1}$$

between the associated toric Deligne-Mumford stacks. By the results of [14] and [20], this map exhibits $\mathfrak{X}(\mathbf{\Sigma})$ as an A-gerbe over $\mathfrak{X}(\mathbf{\Sigma}')$. It can be shown (see *loc. cit*) that this gerbe is a tower of root gerbes.

2.2. **Basics on Gromov-Witten theory.** Gromov-Witten theory for orbifold target spaces is first constructed in symplectic geometry in [8]. In algebraic geometry, the construction is established in [1], [2]. In this Section, we review the main ingredients of orbifold Gromov-Witten theory. We mostly follow the presentation of [24]. More detailed discussions of the basics of orbifold Gromov-Witten theory from the viewpoint of Givental's formalism can be found in e.g. [24], [13].

Let $\mathfrak{X}$ be a smooth proper Deligne-Mumford stack with projective coarse moduli space X. The inertia stack of $\mathfrak{X}$ is defined as

$$I\mathfrak{X} := \mathfrak{X} \times_{\Delta, \mathfrak{X} \times \mathfrak{X}, \Delta} \mathfrak{X},$$

where $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is the diagonal morphism. Additively, the *Chen-Ruan orbifold cohomology* of $\mathfrak{X}$ is defined to be the cohomology of $I\mathfrak{X}$,

$$H^*_{CR}(\mathfrak{X}, \mathbb{C}) := H^*(I\mathfrak{X}, \mathbb{C}).$$

The work [7] equips $H^*_{CR}(\mathfrak{X}, \mathbb{C})$ with a grading called the age grading, a product called Chen-Ruan cup product, and a non-degenerate pairing called the orbifold Poincaré pairing. These are different from the usual ones on $H^*(I\mathfrak{X}, \mathbb{C})$.

Gromov-Witten invariants are constructed based on the moduli spaces of stable maps. Let $\overline{\mathcal{M}}_{g,n}(\mathfrak{X}, d)$ denote the moduli stack of n-pointed genus-g degree-d orbifold stable maps to $\mathfrak{X}$ with sections to all gerbes (see [1, Section 4.5], [24, Section 2.4]). There are evaluation maps at the marked points

$$ev_i : \overline{\mathcal{M}}_{g,n}(\mathfrak{X}, d) \to I\mathfrak{X}, \quad 1 \leq i \leq n,$$

which can be used to pull back classes from $H^*(I\mathfrak{X},\mathbb{C})$. Let $\bar\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}(\mathfrak{X},d),\mathbb{Q})$, $1\le i\le n$ denote the descendant classes, see [24, Section 2.5.1] for more details. Let

$$[\overline{\mathcal{M}}_{g,n}(\mathfrak{X},d)]^w \in H_*(\overline{\mathcal{M}}_{g,n}(\mathfrak{X},d),\mathbb{Q})$$

denote the weighted virtual fundamental class, see [1] and [24, Section 2.5.1] for more details. Given classes $a_1,...,a_n\in H^*(I\mathfrak{X},\mathbb{C})$ and nonnegative integers $k_1,...,k_n$, we define

$$\langle a_1\bar\psi^{k_1},...,a_n\bar\psi^{k_n}\rangle_{g,n,d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathfrak{X},d)]^w}(ev_1^*a_1)\bar\psi_1^{k_1}...(ev_n^*a_n)\bar\psi_n^{k_n}.$$

These are called the *descendant orbifold Gromov-Witten invariants* of $\mathfrak{X}$. We can form generating functions for these invariants. Let $\mathbf{t}=\mathbf{t}(z)=t_0+t_1z+t_2z^2+...\in H^*(I\mathfrak{X})[z]$. Define

$$\langle \mathbf{t},...,\mathbf{t}\rangle_{g,n,d} = \langle \mathbf{t}(\bar\psi),...,\mathbf{t}(\bar\psi)\rangle_{g,n,d} := \sum_{k_1,...,k_n\ge 0}\langle t_{k_1}\bar\psi^{k_1},...,t_{k_n}\bar\psi^{k_n}\rangle_{g,n,d}.$$

The *total descendant potential* is defined to be

$$\mathcal{D}_{\mathfrak{X}}(\mathbf{t}) := \exp\left(\sum_{g\ge 0}\hbar^{g-1}\mathcal{F}^g_{\mathfrak{X}}(\mathbf{t})\right),$$

where

$$\mathcal{F}^g_{\mathfrak{X}}(\mathbf{t}) := \sum_{n\ge 0, d\in \mathrm{Eff}(\mathfrak{X})}\frac{Q^d}{n!}\langle \mathbf{t},...,\mathbf{t}\rangle_{g,n,d}.$$

Here $\hbar$ is a formal variable, and Q^d is an element of the Novikov ring Λ_{nov} which is a certain completion of the group ring $\mathbb{C}[\mathrm{Eff}(\mathfrak{X})]$ of the semi-group $\mathrm{Eff}(\mathfrak{X})$ of effective curve classes (i.e. classes in $H_2(\mathfrak{X},\mathbb{Q})$ represented by images of representable maps from complete stacky curves to $\mathfrak{X}$). $\mathcal{F}^g_{\mathfrak{X}}(\mathbf{t})$ is called the genus-g descendant potential. It is regarded as a Λ_{nov}-valued formal power series in the variables t_k^α where

$$t_k=\sum_\alpha t_k^\alpha\phi_\alpha\in H^*(I\mathfrak{X},\mathbb{C}),\quad k\ge 0.$$

Next we describe Givental's symplectic vector space formalism for genus 0 Gromov-Witten theory [15], [16]. We mainly follow the presentation of [24, Section 3.1]. More details can be found in *loc. cit* as well as and [13] and [9].

Recall (see [24, Definition 2.5.4] that the Novikov ring Λ_{nov} is a topological ring with an additive valuation $v:\Lambda_{nov}\setminus\{0\}\to\mathbb{R}$. Define the space of Λ_{nov}-valued *convergent Laurent series* in z to be

$$\Lambda_{nov}\{z,z^{-1}\} := \left\{\sum_{n\in\mathbb{Z}} r_nz^n \,:\, r_n\in\Lambda_{nov}, v(r_n)\to\infty \text{ as } |n|\to\infty\right\}.$$

Also put

$$\Lambda_{nov}\{z\} := \left\{\sum_{n\ge 0} r_nz^n \,:\, r_n\in\Lambda_{nov}, v(r_n)\to\infty \text{ as } n\to\infty\right\},$$

$$\Lambda_{nov}\{z^{-1}\} := \left\{\sum_{n\le 0} r_nz^n \,:\, r_n\in\Lambda_{nov}, v(r_n)\to\infty \text{ as } -n\to\infty\right\}.$$

Introduce the space of $H^*(I\mathfrak{X}, \Lambda_{nov})$-valued convergent Laurent series,

$$\mathcal{H}^{\mathfrak{X}} := H^*(I\mathfrak{X}, \mathbb{C}) \otimes \Lambda_{nov}\{z, z^{-1}\}.$$

The space $\mathcal{H}^{\mathfrak{X}}$ is equipped with a Λ_{nov}-valued symplectic form define by

$$\Omega_{\mathfrak{X}}(f, g) = \mathrm{Res}_{z=0}(f(-z), g(z))^{\mathfrak{X}}_{orb} dz, \quad \text{for } f, g \in \mathcal{H}^{\mathfrak{X}},$$

where $(-,-)^{\mathfrak{X}}_{orb}$ denotes the orbifold Poincaré pairing on $H^*(I\mathfrak{X}, \mathbb{C})$. Consider the following polarization

$$\begin{aligned} &\mathcal{H}^{\mathfrak{X}} = \mathcal{H}^{\mathfrak{X}}_+ \oplus \mathcal{H}^{\mathfrak{X}}_-, \\ &\mathcal{H}^{\mathfrak{X}}_+ := H^*(I\mathfrak{X}, \mathbb{C}) \otimes \Lambda_{nov}\{z\} \text{ and } \mathcal{H}^{\mathfrak{X}}_- := z^{-1} H^*(I\mathfrak{X}, \mathbb{C}) \otimes \Lambda_{nov}\{z^{-1}\}. \end{aligned} \tag{2}$$

This identifies $\mathcal{H}^{\mathfrak{X}}$ with $\mathcal{H}^{\mathfrak{X}}_+ \oplus \mathcal{H}^{\mathfrak{X}\star}_+$, where $\mathcal{H}^{\mathfrak{X}\star}_+$ is the dual Λ_{nov}-module. We can think of $\mathcal{H}^{\mathfrak{X}}$ as the cotangent bundle $T^*\mathcal{H}^{\mathfrak{X}}_+$. Both $\mathcal{H}^{\mathfrak{X}}_+$ and $\mathcal{H}^{\mathfrak{X}}_-$ are Lagrangian subspaces with respect to $\Omega_{\mathfrak{X}}$.

Let $\{\phi_\nu\} \subset H^*(I\mathfrak{X}, \mathbb{C})$ be an additive basis consisting of homogenerous elements, and let $\{\phi^\mu\} \subset H^*(I\mathfrak{X}, \mathbb{C})$ be the orbifold Poincaré dual basis. In other words, we have $(\phi^\mu, \phi_\nu)^{\mathfrak{X}}_{orb} = \delta^\mu_\nu$. Associated to these bases and the polarization (2) there is a Darboux coordinate system $\{p^\mu_a, q^\nu_b\}$ on $(\mathcal{H}^{\mathfrak{X}}, \Omega_{\mathfrak{X}})$. In these coordinates, a general point in $\mathcal{H}^{\mathfrak{X}}$ takes the form

$$\sum_{a \geq 0} \sum_{\mu} p^\mu_a \phi^\mu (-z)^{-a-1} + \sum_{b \geq 0} \sum_{\nu} q^\nu_b \phi_\nu z^b.$$

Put $p_a = \sum_\mu p^\mu_a \phi^\mu$, $q_b = \sum_\nu q^\nu_b \phi_\nu$, and denote

$$\mathbf{p} = \mathbf{p}(z) := \sum_{k \geq 0} p_k (-z)^{-k-1} = p_0(-z)^{-1} + p_1(-z)^{-2} + \ldots;$$

$$\mathbf{q} = \mathbf{q}(z) := \sum_{k \geq 0} q_k z^k = q_0 + q_1 z + q_2 z^2 + \ldots.$$

For $\mathbf{t}(z) \in \mathcal{H}^{\mathfrak{X}}_+$ introduce a shift $\mathbf{q}(z) = \mathbf{t}(z) - \mathbf{1}z$ called the dilaton shift.

Let *Fock* be the space of $\Lambda_{nov}[[\hbar, \hbar^{-1}]]$-valued formal functions in $\mathbf{t}(z) \in \mathcal{H}^{\mathfrak{X}}_+$. In other words, these are $\Lambda_{nov}[[\hbar, \hbar^{-1}]]$-valued formal power series in variables t^α_k where $t_k = \sum_\alpha t^\alpha_k \phi_\alpha$. We can interpret *Fock* as the space of formal functions on $\mathcal{H}^{\mathfrak{X}}_+$ in the formal neighborhood of $\mathbf{q} = -\mathbf{1}z$. We regard the descendant potential $\mathcal{D}_{\mathfrak{X}}(\mathbf{t})$ as an element in *Fock* via the dilaton shift.

The generating function $\mathcal{F}^0_{\mathfrak{X}}$ of genus-0 orbifold Gromov-Witten invariants defines a formal germ (at the point $-\mathbf{1}z$) of Lagrangian submanifold

$$\mathcal{L}_{\mathfrak{X}} := \{(\mathbf{p}, \mathbf{q}) | \mathbf{p} = d_{\mathbf{q}} \mathcal{F}^0_{\mathfrak{X}}\} \subset \mathcal{H}^{\mathfrak{X}} = T^* \mathcal{H}^{\mathfrak{X}}_+.$$

This is just the graph of the differential of $\mathcal{F}^0_{\mathfrak{X}}$. Equivalently $\mathcal{L}_{\mathfrak{X}}$ is defined by all equations of the form $p^\mu_a = \frac{\partial \mathcal{F}^0_{\mathfrak{X}}}{\partial q^\mu_a}$. By [16, Thereom 1], string and dilaton equations and topological recursion relations imply that $\mathcal{L}_{\mathfrak{X}}$ satisfies the following properties:

Theorem 2.1 (see [9], [24])**.** *$\mathcal{L}_{\mathfrak{X}}$ is the formal germ of a Lagrangian cone with vertex at the origin such that each tangent space T to the cone is tangent to the cone exactly along zT.*

In other words, if N is a formal neighbourhood in $\mathcal{H}$ of the unique geometric point on $\mathcal{L}_{\mathfrak{X}}$, then we have the following statements, valid in the context of formal geometry:

$$\text{(a) } T\cap \mathcal{L}_{\mathfrak{X}} = zT\cap N;$$
$$\text{(b) for each } \mathbf{f}\in zT\cap N, \text{ the tangent space to } \mathcal{L}_{\mathfrak{X}} \text{ at } \mathbf{f} \text{ is } T; \tag{3}$$
$$\text{(c) if } T = T_{\mathbf{f}}\mathcal{L}_{\mathfrak{X}} \text{ then } \mathbf{f}\in zT\cap N.$$

These statements imply that the tangent spaces T of $\mathcal{L}_{\mathfrak{X}}$ are closed under multiplication by z. Moreover, because T/zT is isomorphic to $H^*(I\mathfrak{X},\Lambda_{nov})$, it follows from (3) that $\mathcal{L}_{\mathfrak{X}}$ is the union of the (finite-dimensional) family of germs of (infinite-dimensional) linear subspaces

$$\{zT\cap N | T \text{ is a tangent space of } \mathcal{L}_{\mathfrak{X}}\}.$$

Definition 2.2 (see [24], Definition 3.1.2)**.** Define the J-function $J_{\mathfrak{X}}(t,z)$ to be

$$J_{\mathfrak{X}}(t,z) = z + t + \sum_{n\geq 1, d\in \mathrm{Eff}(\mathfrak{X})} \frac{Q^d}{(n-1)!}\sum_{k\geq 0,\, \alpha} \langle t,...,t,\phi_\alpha\bar{\psi}^k\rangle_{0,n,d}\frac{\phi^\alpha}{z^{k+1}}.$$

This is a formal power series in coordinates t^α of $t=\sum_\alpha t^\alpha\phi_\alpha\in H^*(I\mathfrak{X},\mathbb{C})$ taking values in $\mathcal{H}$. Note that for each $k\geq 0$, the coefficient of the z^{-1-k} term in $J_{\mathfrak{X}}(t,z)$ takes values in $H^*(I\mathfrak{X},\mathbb{C})\otimes\Lambda_{nov}$. The point of $\mathcal{L}_{\mathfrak{X}}$ above $-z+t\in\mathcal{H}_+$ is $J_{\mathfrak{X}}(t,-z)$.

The following result illustrates the importance of the J-function in genus 0 Gromov-Witten theory.

Lemma 2.3 (See [24], Lemma 3.1.3)**.** *The union of the (finite-dimensional) family*

$$t\mapsto zT_{J_{\mathfrak{X}}(t,-z)}\mathcal{L}_{\mathfrak{X}}\cap N, \quad t \text{ in a formal neighborhood of zero in } H^*(I\mathfrak{X},\mathbb{C})\otimes\Lambda_{nov},$$

of germs of linear subspaces is $\mathcal{L}_{\mathfrak{X}}$.

3. Chen-Ruan orbifold cohomology

The purpose of this Section is to study the Chen-Ruan orbifold cohomology ring [7] of a toric gerbe $\mathfrak{X}(\mathbf{\Sigma})$ over $\mathfrak{X}(\mathbf{\Sigma}')$. We show that, in a suitable choice of basis, the Chen-Ruan cohomology $H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}),\mathbb{C})$ is decomposed into a direct sum of several copies of the Chen-Ruan cohomology $H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'),\mathbb{C})$, see Theorem 3.3.

The Chen-Ruan orbifold cohomology ring of a toric Deligne-Mumford stack has been computed[4] in [6]. We recall the answer. Let $M=N^*=\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ be the dual of N. Let $\mathbb{C}[N]^{\mathbf{\Sigma}}$ be the group ring of N, i.e. $\mathbb{C}[N]^{\mathbf{\Sigma}} := \bigoplus_{c\in N}\mathbb{C}y^c$, y is the formal variable. A $\mathbb{Q}$-grading on $\mathbb{C}[N]^{\mathbf{\Sigma}}$ is defined as follows. For $c\in N$, let $\bar{c}\in\overline{N}$ be the image of c under the natural map $N\to\overline{N}$. If $\bar{c}=\sum_{\bar{b}_i\in\sigma(\bar{c})} m_i\bar{b}_i$ where $\sigma(\bar{c})$ is the minimal cone in Σ containing $\bar{c}$ and m_i are nonnegative rational numbers, then we define

$$\deg(y^c) := \sum_{\bar{b}_i\in\sigma(\bar{c})} m_i. \tag{4}$$

[4] Strictly speaking what is computed in [6] is the orbifold Chow ring. However the computation for Chen-Ruan orbifold cohomology ring is identical.

Define the following multiplication on $\mathbb{C}[N]^{\mathbf{\Sigma}}$:

$$y^{c_1}\cdot y^{c_2} := \begin{cases} y^{c_1+c_2} & \text{if there is a cone } \sigma\in\Sigma \text{ such that } \overline{c}_1,\overline{c}_2\in\sigma\,,\\ 0 & \text{otherwise}\,.\end{cases} \tag{5}$$

Let $\mathcal{I}(\mathbf{\Sigma})$ be the ideal in $\mathbb{C}[N]^{\mathbf{\Sigma}}$ generated by the elements $\sum_{i=1}^n \theta(b_i)y^{b_i}, \theta\in M$. Then by [6], Theorem 1.1, there is an isomorphism of $\mathbb{Q}$-graded algebras:

$$H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}),\mathbb{C}) \cong \frac{\mathbb{C}[N]^{\mathbf{\Sigma}}}{\mathcal{I}(\mathbf{\Sigma})}. \tag{6}$$

3.1. **Decomposition of Chen-Ruan cohomology.** Consider the toric gerbe (1) defined by the morphism $\mathbf{\Sigma}\to\mathbf{\Sigma}'$ of stacky fans as in Section 2.1. Since $N=N'\oplus A$, an element $c\in N$ has a unique decomposition $c=(c',\alpha)$ with $c'\in N'$ and $\alpha\in A$. In particular we have

$$b_i=(b_i',\alpha_i)\in N'\oplus A, 1\leq i\leq n.$$

This defines the elements $\alpha_i\in A, 1\leq i\leq n$.

Let $\widehat{A}$ be the set of isomorphism classes of irreducible representations of A. Since A is abelian, the set $\widehat{A}$ is identified with the set of linear characters of A.

Definition 3.1. For $c\in N$, let $\overline{c}\in\overline{N}$ be the image of c under the natural projection $N\to\overline{N}$. Write $\overline{c}=\sum_{\overline{b}_i\subset\sigma(\overline{c})} a_i(c)\overline{b}_i$, where $\sigma(\overline{c})$ is the minimal cone in Σ containing $\overline{c}$ and $a_i(c)$ are non-negative rational numbers. For $[\rho]\in\widehat{A}$ denote by χ_ρ the associated linear character. Define

$$y^{c',\rho} := \frac{1}{|A|}\left(\sum_{\alpha\in A}\chi_\rho(-\alpha)\cdot y^{(c',\alpha)}\right)\cdot\chi_\rho\left(\sum_{i=1}^n a_i(c')\alpha_i\right)\in H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}),\mathbb{C}). \tag{7}$$

Remark 3.2. In the above equation the term $\chi_\rho\left(\sum_{i=1}^n a_i(c)\alpha_i\right)$ is defined to be

$$\prod_{i=1}^n \chi_\rho(\alpha_i)^{a_i(c)}.$$

Note that there exists an integer K such that $a_i(c)\in\mathbb{Z}[\frac{1}{K}]$ for any $c\in N$. The terms $\chi_\rho(\alpha_i)^{a_i(c)}$ are defined by fixing a choice of primitive K-root of 1, say $\zeta_K := \exp(2\pi\sqrt{-1}/K)$.

For each $[\rho]\in\widehat{A}$, let $H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'))_{[\rho]} := H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'),\mathbb{C})$. The direct sum

$$\bigoplus_{[\rho]\in\widehat{A}} H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'))_{[\rho]} \tag{8}$$

inherits a structure of a $\mathbb{Q}$-graded algebra from its summands. For $c'\in N'$ let $y^{c'}_\rho$ denote the element $y^{c'}$ in the summand $H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'))_{[\rho]}$ indexed by $[\rho]$. Then the degree of $y^{c'}_\rho$ is defined to be the degree of $y^{c'}$ in $H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'),\mathbb{C})$. For $c_1',c_2'\in N'$ and $\rho_1,\rho_2\in\widehat{A}$, we have that $y^{c_1'}_{\rho_1}\cdot y^{c_2'}_{\rho_2}=0$ if $[\rho_1]\neq[\rho_2]$. In case $[\rho_1]=[\rho_2]=[\rho]$, the product $y^{c_1'}_{\rho}\cdot y^{c_2'}_{\rho}$ is equal to $y^{c_1'}\cdot y^{c_2'}$.

Theorem 3.3. *The map*

$$\bigoplus_{[\rho]\in\widehat{A}} H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'))_{[\rho]} \longrightarrow H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}),\mathbb{C}),\quad y^{c'}_\rho\mapsto y^{c',\rho}, \tag{9}$$

is an isomorphism of $\mathbb{Q}$-graded algebras.

Proof. We first check that this map preserves grading. By the interpretation (1) of the age-grading in toric case, we can see that for any $\alpha \in A$, the degree of $y^{(c',\alpha)} \in H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}),\mathbb{C})$ coincides with the degree of $y^{c'} \in H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'),\mathbb{C})$. Therefore the degree of $y^{c',\rho} \in H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}),\mathbb{C})$ is the same as the degree of $y^{c'} \in H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'),\mathbb{C})$, which is in turn the same as the degree of $y^{c'}_\rho$ by the definition of grading on the direction sum (8).

To show that this map is an algebra isomorphism, we use the presentation of the Chen-Ruan cohomology ring (6). For $[\rho] \in \widehat{A}$, we write $\mathbb{C}[N']^{\mathbf{\Sigma}'}_{[\rho]} := \mathbb{C}[N']^{\mathbf{\Sigma}'}$ and $\mathcal{I}(\mathbf{\Sigma}')_{[\rho]} := \mathcal{I}(\mathbf{\Sigma})$. In other words,

$$H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'))_{[\rho]} \simeq \mathbb{C}[N']^{\mathbf{\Sigma}'}_{[\rho]}/\mathcal{I}(\mathbf{\Sigma}')_{[\rho]}, \quad \mathbb{C}[N']^{\mathbf{\Sigma}'}_{[\rho]} = \bigoplus_{c'\in N'} \mathbb{C}y^{c'},$$

and $\mathcal{I}(\mathbf{\Sigma}')_{[\rho]}$ is the ideal in $\mathbb{C}[N']^{\mathbf{\Sigma}}_{[\rho]}$ generated by the elements $\sum_{i=1}^n \theta(b'_i)y^{b'_i}$, $\theta \in M'$. Let

$$\mathfrak{M}_A$$

be the square matrix with columns and rows indexed respectively by $\rho \in \widehat{A}$ and $\alpha \in A$ whose (ρ,α)-entry is $\chi_\rho(-\alpha)$. Clearly the matrix $\mathfrak{M}_A$ is invertible. Hence $y^{(c',\alpha)}$ can be expressed as a linear combination of $y^{c',\rho}, \rho \in \widehat{A}$. Hence the map $y^{c'}_\rho \mapsto y^{c',\rho}$ yields a linear isomorphism $\bigoplus_{[\rho]\in\widehat{A}} \mathbb{C}[N']^{\mathbf{\Sigma}'}_{[\rho]} \simeq \mathbb{C}[N]^{\mathbf{\Sigma}}$. It remains to check the following:

Claim 3.1.1. The map $y^{c'}_\rho \mapsto y^{c',\rho}$ is compatible with the product structure defined by (5),

Claim 3.1.2. The map $y^{c'}_\rho \mapsto y^{c',\rho}$ identifies the ideal $\bigoplus_{[\rho]\in\widehat{A}} \mathcal{I}(\mathbf{\Sigma}')_{[\rho]}$ with the ideal $\mathcal{I}(\mathbf{\Sigma})$.

We first check the product structure. Let $c'_1, c'_2 \in N'$ and $[\rho_1],[\rho_2] \in \widehat{A}$. Then we have

$$\begin{aligned} & y^{c'_1,\rho_1} \cdot y^{c'_2,\rho_2} \\ =& \frac{1}{|A|}\left(\sum_{\alpha\in A}\chi_{\rho_1}(-\alpha)\cdot y^{(c'_1,\alpha)}\right)\cdot \chi_{\rho_1}\left(\sum_{i=1}^n a_i(c'_1)\alpha_i\right)\cdot \\ & \cdot \frac{1}{|A|}\left(\sum_{\alpha'\in A}\chi_{\rho_2}(-\alpha')\cdot y^{(c'_2,\alpha')}\right)\cdot \chi_{\rho_2}\left(\sum_{i=1}^n a_i(c'_2)\alpha_i\right). \end{aligned} \tag{10}$$

If $\vec{c}'_1$ and $\vec{c}'_2$ do not lie in the same cone of Σ, then the product $y^{(c'_1,\alpha)}\cdot y^{(c'_2,\alpha')}$ is always 0. Thus in this case $y^{c'_1,\rho_1}\cdot y^{c'_2,\rho_2} = 0$.

Suppose that $\vec{c}'_1$ and $\vec{c}'_2$ lie in the same cone of Σ, then we have

$$y^{(c'_1,\alpha)}\cdot y^{(c'_2,\alpha')} = y^{(c'_1+c'_2,\alpha+\alpha')}.$$

The right-hand side of (10) is equal to

$$\chi_{\rho_1}\left(\sum_{i=1}^n a_i(c'_1)\alpha_i\right)\chi_{\rho_2}\left(\sum_{i=1}^n a_i(c'_2)\alpha_i\right)\cdot\frac{1}{|A|^2}\sum_{\alpha\in A} y^{(c'_1+c'_2,\alpha)}\sum_{\alpha'\in A}\chi_{\rho_1}(\alpha')\chi_{\rho_2}(-\alpha-\alpha'). \tag{11}$$

Since

$$\frac{1}{|A|}\sum_{\alpha'\in A}\chi_{\rho_1}(\alpha')\chi_{\rho_2}(-\alpha-\alpha') = \frac{1}{|A|}\chi_{\rho_2}(-\alpha)\sum_{\alpha'\in A}\chi_{\rho_1}(\alpha')\chi_{\rho_2}(-\alpha') = \chi_{\rho_2}(-\alpha)\delta_{[\rho_1],[\rho_2]},$$

we find that (11) is equal to 0 unless $[\rho_1]=[\rho_2]=:[\rho]$, in which case it is equal to

$$\chi_\rho\left(\sum_{i=1}^n a_i(c_1')\alpha_i\right)\chi_\rho\left(\sum_{i=1}^n a_i(c_2')\alpha_i\right)\cdot\frac{1}{|A|}\sum_{\alpha\in A}\chi_\rho(-\alpha)y^{(c_1'+c_2',\alpha)}. \tag{12}$$

Note also that $a_i(c_1'+c_2')=a_i(c_1')+a_i(c_2')$ since $\bar{c}_1'$ and $\bar{c}_2'$ lie in the same cone of Σ. Hence

$$\chi_\rho\left(\sum_{i=1}^n a_i(c_1')\alpha_i\right)\chi_\rho\left(\sum_{i=1}^n a_i(c_2')\alpha_i\right) = \chi_\rho\left(\sum_{i=1}^n a_i(c_1'+c_2')\alpha_i\right),$$

and (12) is equal to $y^{c_1'+c_2',\rho}$.

In summary, $y^{c_1',\rho_1}\cdot y^{c_2',\rho_2}$ is equal to $y^{c_1'+c_2',\rho}$ if $[\rho_1]=[\rho_2]=[\rho]$ and $\bar{c}_1',\bar{c}_2'$ lie in the same cone of Σ. Otherwise $y^{c_1',\rho_1}\cdot y^{c_2',\rho_2}=0$. Clearly this agrees with the product $y_{\rho_1}^{c_1'}\cdot y_{\rho_2}^{c_2'}$ under the map $y_\rho^{c'}\mapsto y^{c',\rho}$. This proves Claim 3.1.1.

We now turn to Claim 3.1.2. Note that $M=\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})=\mathrm{Hom}_{\mathbb{Z}}(N',\mathbb{Z})=M'$. Let $\theta\in M$. Then we have $\theta(b_i)=\theta(b_i')$. For any $\rho\in\widehat{A}$, the map $y_\rho^{c'}\mapsto y^{c',\rho}$ takes the relation

$$\sum_{i=1}^n\theta(b_i')y_\rho^{b_i'}=0$$

to the relation

$$\sum_{i=1}^n\theta(b_i')y^{b_i',\rho}=0. \tag{13}$$

We may rewrite the left-hand side of (13) as follows:

$$\begin{aligned}\sum_{i=1}^n\theta(b_i')y^{b_i',\rho} &= \sum_{i=1}^n\theta(b_i')\frac{1}{|A|}\left(\sum_{\alpha\in A}\chi_\rho(-\alpha)\cdot y^{(b_i',\alpha)}\right)\cdot\chi_\rho(\alpha_i)\\ &= \sum_{i=1}^n\theta(b_i')\frac{1}{|A|}\left(\sum_{\alpha\in A}\chi_\rho(-\alpha+\alpha_i)\cdot y^{(b_i',\alpha)}\right)\\ &= \sum_{i=1}^n\theta(b_i')\frac{1}{|A|}\left(\sum_{\alpha\in A}\chi_\rho(-\alpha)\cdot y^{(b_i',\alpha+\alpha_i)}\right)\\ &= \frac{1}{|A|}\sum_{\alpha\in A}\chi_\rho(-\alpha)\left(y^{(0,\alpha)}\sum_{i=1}^n\theta(b_i)y^{b_i}\right),\end{aligned} \tag{14}$$

where in the last equality we use the fact that $y^{(b_i',\alpha+\alpha_i)}=y^{(b_i',\alpha_i)+(0,\alpha)}$, $(b_i',\alpha_i)=b_i$, and $\theta(b_i')=\theta(b_i)$. Since the matrix $\mathfrak{M}_A$ is invertible, we find that the collection of relations (13) for all $\rho\in\widehat{A}$ is equivalent to the relations

$$y^{(0,\alpha)}\sum_{i=1}^n\theta(b_i)y^{b_i}=0,\quad \alpha\in A.$$

This is clearly equivalent to the relation

$$\sum_{i=1}^{n}\theta(b_i)y^{b_i}=0. \tag{15}$$

Claim 3.1.2 is proved. This concludes the proof of the Theorem. □

We now describe the compatibility of the map (9) with orbifold Poincaré pairings. Let $(-,-)_{orb}^{\mathfrak{X}(\mathbf{\Sigma}'),[\rho]}:=(-,-)_{orb}^{\mathfrak{X}(\mathbf{\Sigma}')}$ denote the orbifold Poincaré pairing on the summand $H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'))_{[\rho]}=H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'),\mathbb{C})$ of (8) indexed by $[\rho]\in\widehat{A}$.

Proposition 3.4. *The map (9) identifies the orbifold Poincaré pairing $(-,-)_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}$ on $H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}),\mathbb{C})$ with the following non-degenerate pairing on the direct sum (8):*

$$\bigoplus_{[\rho]\in\widehat{A}}\frac{1}{|A|^2}(-,-)_{orb}^{\mathfrak{X}(\mathbf{\Sigma}'),[\rho]}. \tag{16}$$

Proof. Let $c_1',c_2'\in N'$ and $[\rho_1],[\rho_2]\in\widehat{A}$. To prove the Proposition it suffices to prove the following two statements:

$$(y^{c_1',\rho_1},y^{c_2',\rho_2})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}=0\ \text{ if } [\rho_1]\neq[\rho_2]; \tag{17}$$

$$(y^{c_1',\rho},y^{c_2',\rho})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}=\frac{1}{|A|^2}(y^{c_1'},y^{c_2'})_{orb}^{\mathfrak{X}(\mathbf{\Sigma}')}\ \text{ if } [\rho_1]=[\rho_2]=[\rho]. \tag{18}$$

Properties of the Chen-Ruan cup product imply that

$$(y^{c_1',\rho_1},y^{c_2',\rho_2})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}=(y^{c_1',\rho_1}\cdot y^{c_2',\rho_2},\mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma})})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})},$$

where $\mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma})}\in H^0(\mathfrak{X}(\mathbf{\Sigma}),\mathbb{C})$ is the unit class. (17) now follows from the fact (see the proof of Claim 3.1.1) that $y^{c_1',\rho_1}\cdot y^{c_2',\rho_2}=0$ when $[\rho_1]\neq[\rho_2]$.

Now suppose that $[\rho_1]=[\rho_2]=[\rho]$. Since $(y^{c_1'},y^{c_2'})_{orb}^{\mathfrak{X}(\mathbf{\Sigma}')}=(y^{c_1'}\cdot y^{c_2'},\mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma}')})_{orb}^{\mathfrak{X}(\mathbf{\Sigma}')}$, where $\mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma}')}\in H^0(\mathfrak{X}(\mathbf{\Sigma}'),\mathbb{C})$ is the unit class, it follows that both sides of (18) are 0 unless $\vec{c}_1',\vec{c}_2'$ belong to the same cone of Σ'. Therefore we may assume that $\vec{c}_1',\vec{c}_2'$ belong to the same cone of Σ'. Since $y^{c_1',\rho}\cdot y^{c_2',\rho}=y^{c_1'+c_2',\rho}$ and $y^{c_1'}\cdot y^{c_2'}=y^{c_1'+c_2'}$, to prove (18) it remains to show the following statement:

$$(y^{c',\rho},\mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma})})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}=\frac{1}{|A|^2}(y^{c'},\mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma}')})_{orb}^{\mathfrak{X}(\mathbf{\Sigma}')},\quad c'\in N'. \tag{19}$$

By (7), the left-hand side of (19) is

$$(y^{c',\rho},\mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma})})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}=\frac{1}{|A|}\left(\sum_{\alpha\in A}\chi_\rho(-\alpha)\cdot(y^{(c',\alpha)},\mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma})})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}\right)\cdot\chi_\rho\left(\sum_{i=1}^{n}a_i(c')\alpha_i\right).$$

In order for $(y^{(c',\alpha)},\mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma})})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}$ to be non-zero, the support of the class $y^{(c',\alpha)}$ must be contained in the untwisted sector $\mathfrak{X}(\mathbf{\Sigma})$. In view of the identification of $H^*(\mathfrak{X}(\mathbf{\Sigma}),\mathbb{C})$ in [6, Lemma 5.1], this means that $(c',\alpha)=\sum_i n_ib_i\in N$ for some $n_i\in\mathbb{N}_{\geq 0}$. Moreover,

$$(c',\alpha)=\sum_i n_ib_i=\sum_i n_i(b_i',\alpha_i)=(\sum_i n_ib_i',\sum_i n_i\alpha_i).$$

Comparing with $\bar{c}' = \sum_i a_i(c')\bar{b}'_i$ and note that $\bar{b}_i = \bar{b}'_i$, we find

$$c' = \sum_i n_i b'_i, \quad a_i(c') = n_i, \quad \alpha = \sum_i n_i\alpha_i = \sum_i a_i(c')\alpha_i.$$

Therefore

$$\begin{aligned}&(y^{c',\rho}, \mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma})})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}\\ =&\frac{1}{|A|}\chi_\rho\left(-\sum_i n_i\alpha_i\right)\cdot (y^{\sum_i n_i b_i}, \mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma})})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}\cdot \chi_\rho\left(\sum_{i=1}^n a_i(c')\alpha_i\right)\\ =&\frac{1}{|A|}(y^{\sum_i n_i b_i}, \mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma})})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}.\end{aligned}$$

Since the map $\mathfrak{X}(\mathbf{\Sigma}) \to \mathfrak{X}(\mathbf{\Sigma}')$ is of degree $1/|A|$, and the classes $y^{b'_i} \in H^*(\mathfrak{X}(\mathbf{\Sigma}'), \mathbb{C})$ pull back to $y^{b_i} \in H^*(\mathfrak{X}(\mathbf{\Sigma}), \mathbb{C})$ via $\mathfrak{X}(\mathbf{\Sigma}) \to \mathfrak{X}(\mathbf{\Sigma}')$, it follows that

$$\begin{aligned}&(y^{\sum_i n_i b_i}, \mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma})})_{orb}^{\mathfrak{X}(\mathbf{\Sigma})}\\ =&\int_{\mathfrak{X}(\mathbf{\Sigma})} y^{\sum_i n_i b_i}\\ =&\frac{1}{|A|}\int_{\mathfrak{X}(\mathbf{\Sigma}')} y^{\sum_i n_i b'_i}\\ =&\frac{1}{|A|}(y^{\sum_i n_i b'_i}, \mathbf{1}_{\mathfrak{X}(\mathbf{\Sigma}')})_{orb}^{\mathfrak{X}(\mathbf{\Sigma}')},\end{aligned}$$

which proves (19). This concludes the proof of the Proposition. □

For each $[\rho] \in \widehat{A}$ let $(\mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma}'),\rho}, \Omega_{\mathfrak{X}(\mathbf{\Sigma}'),\rho})$ be the symplectic space $(\mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma}')}, \frac{\Omega_{\mathfrak{X}(\mathbf{\Sigma}')}}{|A|^2})$. Let $\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma}'),\rho} \subset \mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma}'),\rho}$ denote the Lagrangian cone $\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma}')}$. Then Theorem 3.3 and Proposition 3.4 imply

Theorem 3.5. *The isomorphism (9) yields an isomorphism of symplectic vector spaces*

$$\bigoplus_{[\rho]\in\widehat{A}} (\mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma}'),\rho}, \Omega_{\mathfrak{X}(\mathbf{\Sigma}'),\rho}) \xrightarrow{\simeq} (\mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma})}, \Omega_{\mathfrak{X}(\mathbf{\Sigma})}). \tag{20}$$

Here we identify the ground ring $\Lambda_{nov}(\mathfrak{X}(\mathbf{\Sigma}))$ for $\mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma})}$ with the ground ring $\Lambda_{nov}(\mathfrak{X}(\mathbf{\Sigma}'))$ for $\mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma}'),\rho}$ via $Q_i \mapsto Q_i, 1 \leq i \leq n$.

Moreover, the isomorphism (20) identifies the direct sum $\bigoplus_{[\rho]\in\widehat{A}} \mathcal{H}_+^{\mathfrak{X}(\mathbf{\Sigma}'),\rho}$ with $\mathcal{H}_+^{\mathfrak{X}(\mathbf{\Sigma})}$.

The following result will be used in the next Section.

Lemma 3.6. *Let $\rho \in \widehat{A}$. Then for $v' \in N'$ we have*

$$y^{v',\rho}\cdot y^{b_i} = y^{v',\rho}\cdot y^{b'_i,\rho}.$$

Proof. We may assume that $\bar{v}'$ and $\bar{b}_i'$ lie in the same cone. Then

$$\begin{aligned}
&y^{v',\rho}\cdot y^{b_i}\\
=&\frac{1}{|A|}\left(\sum_{\alpha\in A}\chi_\rho(-\alpha)\cdot y^{(v',\alpha)}\right)\cdot\chi_\rho\left(\sum_{i=1}^n a_i(v')\alpha_i\right)\cdot y^{b_i}\\
=&\frac{1}{|A|}\left(\sum_{\alpha\in A}\chi_\rho(-\alpha)\cdot y^{(v'+b_i',\alpha+\alpha_i)}\right)\cdot\chi_\rho\left(\sum_{i=1}^n a_i(v')\alpha_i\right)\\
=&\frac{1}{|A|}\left(\sum_{\alpha\in A}\chi_\rho(-\alpha)\cdot y^{(v'+b_i',\alpha)}\right)\cdot\chi_\rho\left(\sum_{i=1}^n a_i(v')\alpha_i\right)\chi_\rho(\alpha_i).
\end{aligned}\tag{21}$$

$$\begin{aligned}
&y^{v',\rho}\cdot y^{b_i',\rho}\\
=&\frac{1}{|A|}\left(\sum_{\alpha_1\in A}\chi_\rho(-\alpha_1)\cdot y^{(v',\alpha_1)}\right)\cdot\chi_\rho\left(\sum_{i=1}^n a_i(v')\alpha_i\right)\cdot\\
&\cdot\frac{1}{|A|}\left(\sum_{\alpha_2\in A}\chi_\rho(-\alpha_2)\cdot y^{(b_i',\alpha_2)}\right)\cdot\chi_\rho\left(\alpha_i\right)\\
=&\frac{1}{|A|^2}\left(\sum_{\alpha_1,\alpha_2\in A}\chi_\rho(-\alpha_1-\alpha_2)y^{(v'+b_i',\alpha_1+\alpha_2)}\right)\cdot\chi_\rho\left(\sum_{i=1}^n a_i(v')\alpha_i\right)\cdot\chi_\rho\left(\alpha_i\right)\\
=&\frac{1}{|A|}\left(\sum_{\alpha\in A}\chi_\rho(-\alpha)\cdot y^{(v'+b_i',\alpha)}\right)\cdot\chi_\rho\left(\sum_{i=1}^n a_i(v')\alpha_i\right)\chi_\rho(\alpha_i).
\end{aligned}\tag{22}$$

This concludes the proof. □

4. Genus 0 Gromov-Witten theory

In this Section we study Gromov-Witten theory of a toric gerbe $\mathfrak{X}(\mathbf{\Sigma})$ over a toric Deligne-Mumford stack $\mathfrak{X}(\mathbf{\Sigma}')$. Our main result is an explicit comparison of Gromov-Witten theory of $\mathfrak{X}(\mathbf{\Sigma})$ and that of $\mathfrak{X}(\mathbf{\Sigma}')$, which can be interpreted as a Gromov-Witten theoretic version of the decomposition conjecture. The main tool is a detailed calculation of Gromov-Witten invariants of toric stacks [10].

4.1. **Genus zero Gromov-Witten invariants of toric stacks.** In this subsection we present a summary of the results of [10]. Consider a toric Deligne-Mumford stack $\mathfrak{X}(\mathbf{\Sigma})$ defined by a stacky fan $(N,\Sigma,\beta:\mathbb{Z}^n\to N)$. As in Section 2.1, the fan map $\beta:\mathbb{Z}^n\to N$ is given by elements $\{b_1,...,b_n\}\subset N$. In other words, let $e_i, 1\le i\le n$ be the standard basis of $\mathbb{Z}^n$. Then $\beta(e_i)=b_i$.

Let

$$S:=\{s_j|1\le j\le m\}\subset N$$

be a subset of N. The *S-extended stacky fan* (see [19]) is given by the same group N, the same fan Σ, and the following fan map

$$\beta^S:\mathbb{Z}^{n+m}\to N;\quad e_i\mapsto b_i, 1\le i\le n,\quad e_j'\mapsto s_j, 1\le j\le m.\tag{23}$$

Here we write $\mathbb{Z}^{n+m}=\mathbb{Z}^n\oplus\mathbb{Z}^m$ and let $\{e_i|1\le i\le n\}$ be the standard basis of $\mathbb{Z}^n$, and let $\{e_j'|1\le j\le m\}$ be the standard basis of $\mathbb{Z}^m$. It is known [19] that the stack associated to the S-extended stacky fan (N, Σ, β^S) is isomorphic to the stack $\mathfrak{X}(\mathbf{\Sigma})$.

Let $\mathbb{L}^S$ be the kernel of $\beta^S : \mathbb{Z}^{n+m} \to N$. Applying Gale duality to the S-extended fan sequence $0 \to \mathbb{L}^S \to \mathbb{Z}^{n+m} \to N$ give s the S-extended divisor sequence,

$$0 \to N^* \to (\mathbb{Z}^*)^{n+m} \to \mathbb{L}^{S\vee}.$$

If σ is a cone of Σ. Define

$$C^S_\sigma := \left\{ \sum_{1\leq i\leq n, e_i\notin\sigma} r_i D^S(e_i^*) + \sum_j r'_j D^S({e^*}'_j) | r_i, r'_j \geq 0 \right\} \subset \mathbb{L}^{S\vee}.$$

Put $\mathrm{NE}^S_\sigma := C^{S\vee}_\sigma$. Define the S-extended Mori cone to be $\mathrm{NE}^S(\mathfrak{X}(\mathbf{\Sigma})) := \sum_{\sigma\in\Sigma} \mathrm{NE}^S_\sigma$.

We next introduce a number of combinatorial objects associated to the S-extended stacky fan. Let σ be a cone of Σ. Denote by $\Lambda^S_\sigma \subset \mathbb{L}^S_{\mathbb{Q}}$ the subset consisting of elements

$$\lambda = \sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^m \lambda'_j e'_j \in \mathbb{Q}^{n+m}$$

such that $\lambda'_j \in \mathbb{Z}, 1 \leq j \leq m$ and $\lambda_i \in \mathbb{Z}$ if $e_j \notin \sigma$. Set $\Lambda^S := \cup_{\sigma\in\Sigma}\Lambda^S_\sigma$.

Define the (outgoing) reduction function to be

$$v^S : \Lambda^S \to \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma})), \quad v^S(\lambda) := \sum_{i=1}^n \lceil \lambda_i \rceil b_i + \sum_{j=1}^m \lceil \lambda'_j \rceil s_j. \tag{24}$$

Now we define

$$\begin{aligned} \Lambda E^S(\mathfrak{X}(\mathbf{\Sigma})) &:= \Lambda^S \cap \mathrm{NE}^S(\mathfrak{X}(\mathbf{\Sigma})), \\ \Lambda E^S_v &:= \{\lambda \in \Lambda E^S(\mathfrak{X}(\mathbf{\Sigma})) | v^S(\lambda) = v\}. \end{aligned} \tag{25}$$

We can now write down the *S-extended I-function* of $\mathfrak{X}(\mathbf{\Sigma})$:

$$\begin{aligned} I^S_{\mathfrak{X}(\mathbf{\Sigma})}(Q, t, z) := z e^{\sum_{i=1}^n y^{b_i} \frac{\log Q_i}{z}} & \sum_{v\in\mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}))} \sum_{\lambda\in\Lambda E^S_v} \prod_{i=1}^n Q_i^{\lambda_i} \prod_{j=1}^m t_j^{\lambda'_j} y^v \times \\ & \times \prod_{i=1}^n \frac{\prod_{\langle b\rangle = v_i, b\leq 0}(y^{b_i} + bz)}{\prod_{\langle b\rangle = v_i, b\leq\lambda_i}(y^{b_i} + bz)} \prod_{j=1}^m \frac{\prod_{b\leq 0}(bz)}{\prod_{b\leq\lambda'_j}(bz)}. \end{aligned} \tag{26}$$

Here for $v \in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}))$, we write $\bar{v} = \sum_i v_i \bar{b}_i$.

One of the main results of [10] is the following

Theorem 4.1. *The extended I-function $I^S_{\mathfrak{X}(\mathbf{\Sigma})}$ is contained in the Lagrangian cone $\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma})}$.*

For a suitable choice of S, it is easy to check that the S-extended I-function $I^S_{\mathfrak{X}(\mathbf{\Sigma})}$ is a slice of the cone $\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma})}$ transverse to the ruling. We can then use $I^S_{\mathfrak{X}(\mathbf{\Sigma})}$ to determine the whole cone $\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma})}$, hence the full genus 0 Gromov-Witten theory of $\mathfrak{X}(\mathbf{\Sigma})$.

4.2. **Comparison of genus zero invariants.** In this subsection we apply the results of [10] to toric gerbes. Consider the toric gerbe (1) defined by the morphism $\mathbf{\Sigma} \to \mathbf{\Sigma}'$ of stacky fans as in Section 2.1. Our strategy is to suitably choose the extension set S for $\mathfrak{X}(\mathbf{\Sigma})$ and S' for $\mathfrak{X}(\mathbf{\Sigma}')$, so that we can explicitly compare the extended I-functions $I^S_{\mathfrak{X}(\mathbf{\Sigma})}$ and $I^{S'}_{\mathfrak{X}(\mathbf{\Sigma}')}$.

Let

$$S' := \{s'_j | 1 \leq j \leq m\} \subset N'$$

be a subset. We define an S'-extended fan map $\mathbb{Z}^{n+m} \to N'$ as follows. Write $\mathbb{Z}^{n+m} = \mathbb{Z}^n \oplus \mathbb{Z}^m$ and let $e_i, 1 \leq i \leq n$ be the standard basis of $\mathbb{Z}^n$ and $e'_j, 1 \leq j \leq m$ the standard basis of $\mathbb{Z}^m$. The map $\mathbb{Z}^{n+m} \to N'$ is defined by

$$e_i \mapsto b'_i, 1 \leq i \leq n; \quad e'_j \mapsto s'_j, 1 \leq j \leq m.$$

Define a subset of $N = N' \oplus A$ by

$$S := \{(s'_j, \alpha) | 1 \leq j \leq m, \alpha \in A\} \subset N.$$

We define a S-extended fan map $\mathbb{Z}^{n+|A|m} \to N$ as follows. Write $\mathbb{Z}^{n+|A|m} = \mathbb{Z}^n \oplus \mathbb{Z}^{|A|m}$ and let $e_i, 1 \leq i \leq n$ be the standard basis of $\mathbb{Z}^n$ and $e'_{j,\alpha}, 1 \leq j \leq m, \alpha \in A$ the standard basis of $\mathbb{Z}^{|A|m}$. The map $\mathbb{Z}^{n+|A|m} \to N$ is defined by

$$e_i \mapsto b_i, 1 \leq i \leq n; \quad e'_{j,\alpha} \mapsto (s'_j, \alpha), 1 \leq j \leq m, \alpha \in A.$$

These extended fan maps fit into the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^{n+|A|m} & \longrightarrow & N \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}^{n+m} & \longrightarrow & N', \end{array}$$

where the vertical map $N \to N'$ is the natural projection, and the vertical map $\mathbb{Z}^{n+|A|m} \to \mathbb{Z}^{n+m}$ is given by

$$e_i \mapsto e_i, 1 \leq i \leq n; \quad e'_{j,\alpha} \mapsto e'_j, 1 \leq j \leq m, \forall \alpha \in A.$$

Let $v' \in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}'))$ and $a \in A$, then $(v', a) \in N' \oplus A = N$ is contained in $\mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}))$. Let $\sigma(v')$ be the minimal cone containing v'. Write $\bar{v}' = \sum_{b'_i \in \sigma(v')} v_i \bar{b}'_i$ with $v_i \in \mathbb{Q}$.

An element $\lambda \in \Lambda E^S_{(v',a)}$ is of the form

$$\lambda = \sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^s \sum_{\alpha \in A} \lambda_{j,\alpha} e'_{j,\alpha} \in \mathbb{L}^S_{\mathbb{Q}} \tag{27}$$

such that

1. if e_i is not in the minimal cone of v', then $\lambda_i \in \mathbb{Z}$;
2. $\lambda_{j,\alpha} \in \mathbb{Z}_{\geq 0}$.

Following [10] the S-extended I-function of $\mathfrak{X}(\mathbf{\Sigma})$ can be written as follows:

(28)

$$\begin{aligned}
& e^{-\sum_{i=1}^n y^{b_i}\log Q_i/z} I^S_{\mathfrak{X}(\mathbf{\Sigma})}(Q,t,z)\\
=& z\sum_{(v',a)\in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}))}\sum_{\lambda\in\Lambda E^S_{(v',a)}}\prod_{i=1}^n Q_i^{\lambda_i}\prod_{j=1}^m\prod_{\alpha\in A} t_{j,\alpha}^{\lambda_{j,\alpha}}\\
& \times y^{(v',a)}\prod_{i=1}^n\frac{\prod_{\langle b\rangle=v_i,b\leq 0}(y^{b_i}+bz)}{\prod_{\langle b\rangle=v_i,b\leq\lambda_i}(y^{b_i}+bz)}\prod_{j=1}^m\prod_{\alpha\in A}\frac{1}{\prod_{b=1}^{\lambda_{j,\alpha}}(bz)}\\
=& z\sum_{(v',a)\in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}))}\sum_{\lambda\in\Lambda E^S_{(v',a)}}\prod_{i=1}^n Q_i^{\lambda_i}\prod_{j=1}^m\prod_{\alpha\in A} t_{j,\alpha}^{\lambda_{j,\alpha}}\\
& \left(\sum_{\rho\in\widehat{A}}\chi_\rho(a)y^{v',\rho}\chi_\rho(-\sum_{i=1}^n a_i(v')\alpha_i)\right)\prod_{i=1}^n\frac{\prod_{\langle b\rangle=v_i,b\leq 0}(y^{b_i}+bz)}{\prod_{\langle b\rangle=v_i,b\leq\lambda_i}(y^{b_i}+bz)}\prod_{j=1}^m\prod_{\alpha\in A}\frac{1}{\lambda_{j,\alpha}!z^{\lambda_{j,\alpha}}}\\
=& z\sum_{(v',a)\in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}))}\sum_{\lambda\in\Lambda E^S_{(v',a)}}\prod_{i=1}^n Q_i^{\lambda_i}\prod_{j=1}^m\prod_{\alpha\in A}\frac{1}{\lambda_{j,\alpha}!}\left(\frac{t_{j,\alpha}}{z}\right)^{\lambda_{j,\alpha}}\\
& \left(\sum_{\rho\in\widehat{A}}\chi_\rho(a)y^{v',\rho}\chi_\rho(-\sum_{i=1}^n a_i(v')\alpha_i)\right)\prod_{i=1}^n\frac{\prod_{\langle b\rangle=v_i,b\leq 0}(y^{b_i}+bz)}{\prod_{\langle b\rangle=v_i,b\leq\lambda_i}(y^{b_i}+bz)}\\
=& \sum_{\rho\in\widehat{A}}\left(z\sum_{v'\in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}')),a\in A}\sum_{\lambda\in\Lambda E^S_{(v',a)}}\prod_{i=1}^n Q_i^{\lambda_i}\prod_{j=1}^m\prod_{\alpha\in A}\frac{1}{\lambda_{j,\alpha}!}\left(\frac{t_{j,\alpha}}{z}\right)^{\lambda_{j,\alpha}}\right.\\
& \qquad\left.\times\chi_\rho(a)\chi_\rho(-\sum_{i=1}^n a_i(v')\alpha_i)y^{v',\rho}\frac{\prod_{\langle b\rangle=v_i,b\leq 0}(y^{b_i}+bz)}{\prod_{\langle b\rangle=v_i,b\leq\lambda_i}(y^{b_i}+bz)}\right),
\end{aligned}$$

where we have used the substitution

$$y^{(v',a)}=\sum_{\rho\in\widehat{A}}\chi_\rho(a)y^{v',\rho}\chi_\rho(-\sum_{i=1}^n a_i(v')\alpha_i) \tag{29}$$

obtained by inverting (7).

Since $\lambda\in\Lambda E^S_{(v',a)}$, we have

$$a=\sum_{i=1}^n\lceil\lambda_i\rceil\alpha_i+\sum_{\alpha\in A}(\sum_{j=1}^m\lceil\lambda_{j,\alpha}\rceil)\alpha;\quad v'=\sum_{i=1}^n\lceil\lambda_i\rceil b'_i+\sum_{j=1}^m(\sum_{\alpha\in A}\lceil\lambda_{j,\alpha}\rceil)s'_j. \tag{30}$$

Note that

$$\bar{v}'=\sum_{i=1}^n\lceil\lambda_i\rceil\bar{b}'_i+\sum_{j=1}^m(\sum_{\alpha\in A}\lceil\lambda_{j,\alpha}\rceil)\bar{s}'_j=\sum_{i=1}^n\langle-\lambda_i\rangle\,\bar{b}'_i. \tag{31}$$

Since by definition

$$\bar{v}'=\sum_{\bar{b}_i\in\sigma(\bar{v}')}a_i(v')\bar{b}'_i,$$

we have

$$a_i(v') = \langle -\lambda_i \rangle .$$

We compute

$$\begin{aligned} a - \sum_{i=1}^n a_i(v')\alpha_i &= \sum_{i=1}^n (\lceil \lambda_i \rceil - \langle -\lambda_i \rangle)\alpha_i + \sum_{\alpha\in A}(\sum_{j=1}^m \lceil \lambda_{j,\alpha} \rceil)\alpha \\ &= \sum_{i=1}^n \lambda_i\alpha_i + \sum_{\alpha\in A}(\sum_{j=1}^m \lambda_{j,\alpha})\alpha. \end{aligned} \tag{32}$$

So

$$\begin{aligned} &\prod_{i=1}^n Q_i^{\lambda_i} \prod_{j=1}^m \prod_{\alpha\in A} \frac{1}{\lambda_{j,\alpha}!}\left(\frac{t_{j,\alpha}}{z}\right)^{\lambda_{j,\alpha}} \chi_\rho(a)\chi_\rho(-\sum_{i=1}^n a_i(v')\alpha_i) \\ =&\prod_{i=1}^n (Q_i\chi_\rho(\alpha_i))^{\lambda_i} \prod_{j=1}^m \prod_{\alpha\in A} \frac{1}{\lambda_{j,\alpha}!}\left(\frac{t_{j,\alpha}\chi_\rho(\alpha)}{z}\right)^{\lambda_{j,\alpha}}. \end{aligned} \tag{33}$$

This allows us to rewrite the last expression of (28) as

$$\begin{aligned} &\sum_{\rho\in\widehat{A}} z \sum_{v'\in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}')), a\in A}\ \sum_{\lambda\in\Lambda E^S_{(v',a)}} \prod_{i=1}^n (Q_i\chi_\rho(\alpha_i))^{\lambda_i}\times \\ &\times \prod_{j=1}^m \prod_{\alpha\in A} \frac{1}{\lambda_{j,\alpha}!}\left(\frac{t_{j,\alpha}\chi_\rho(\alpha)}{z}\right)^{\lambda_{j,\alpha}} y^{v',\rho}\frac{\prod_{\langle b\rangle=v_i, b\leq 0}(y^{b_i}+bz)}{\prod_{\langle b\rangle=v_i, b\leq \lambda_i}(y^{b_i}+bz)}. \end{aligned} \tag{34}$$

Now we fix a representation $\rho\in\widehat{A}$ and consider the summand

$$\begin{aligned} &z \sum_{v'\in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}')), a\in A}\ \sum_{\lambda\in\Lambda E^S_{(v',a)}} \prod_{i=1}^n (Q_i\chi_\rho(\alpha_i))^{\lambda_i}\times \\ &\times \prod_{j=1}^m \prod_{\alpha\in A} \frac{1}{\lambda_{j,\alpha}!}\left(\frac{t_{j,\alpha}\chi_\rho(\alpha)}{z}\right)^{\lambda_{j,\alpha}} y^{v',\rho}\frac{\prod_{\langle b\rangle=v_i, b\leq 0}(y^{b_i}+bz)}{\prod_{\langle b\rangle=v_i, b\leq \lambda_i}(y^{b_i}+bz)}. \end{aligned} \tag{35}$$

Given $\lambda\in\Lambda E^S_{(v',a)}$ as in (27), by (30), we have

$$\sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^m (\sum_{\alpha\in A}\lambda_{j,\alpha})e'_j \in \Lambda E^{S'}_{v'} \subset \mathbb{L}^{S'}_{\mathbb{Q}}. \tag{36}$$

Denote a general element $\lambda'\in\Lambda E^{S'}_{v'}$ by $\lambda'=\sum_{i=1}^n\lambda_i e_i+\sum_{j=1}^m \lambda'_j e'_j$. Note that in (35) we sum over all elements of A. Also note that by Lemma 3.6,

$$y^{v',\rho}\cdot y^{b_i} = y^{v',\rho}\cdot y^{b'_i,\rho}.$$

Thus (35) can be written as

(37)

$$\begin{aligned} &z \sum_{v'\in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}'))}\ \sum_{\lambda'\in\Lambda E^{S'}_{v'}} \prod_{i=1}^n (Q_i\chi_\rho(\alpha_i))^{\lambda_i} \\ &\sum_{\lambda_{j,\alpha}\in\mathbb{Z}_{\geq 0}, \lambda'_j=\sum_{\alpha\in A}\lambda_{j,\alpha}} \prod_{j=1}^m \prod_{\alpha\in A} \frac{1}{\lambda_{j,\alpha}!}\left(\frac{t_{j,\alpha}\chi_\rho(\alpha)}{z}\right)^{\lambda_{j,\alpha}} y^{v',\rho}\frac{\prod_{\langle b\rangle=v_i, b\leq 0}(y^{b'_i,\rho}+bz)}{\prod_{\langle b\rangle=v_i, b\leq \lambda_i}(y^{b'_i,\rho}+bz)}. \end{aligned}$$

Applying the polynomial theorem to one j at a time, we find that (37) is equal to

$$z\sum_{v'\in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}'))}\sum_{\lambda'\in\Lambda E^{S'}_{v'}}\prod_{i=1}^{n}(Q_i\chi_\rho(\alpha_i))^{\lambda_i}\prod_{j=1}^{m}\frac{1}{\lambda'_j!}\left(\frac{\sum_{\alpha\in A}t_{j,\alpha}\chi_\rho(\alpha)}{z}\right)^{\lambda'_j}y^{v',\rho}\times \times\frac{\prod_{\langle b\rangle=v_i,b\le 0}(y^{b'_i,\rho}+bz)}{\prod_{\langle b\rangle=v_i,b\le\lambda_i}(y^{b'_i,\rho}+bz)}. \tag{38}$$

By the change of variables

$$t_{j,\rho}=(\sum_{\alpha\in A}t_{j,\alpha}\chi_\rho(\alpha))\chi_\rho(-\sum_{i=1}^{n}a_i(s'_j)\alpha_i), \tag{39}$$

we may rewrite (38) as

(40)

$$\begin{aligned}&z\sum_{v'\in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}'))}\sum_{\lambda'\in\Lambda E^{S'}_{v'}}\prod_{i=1}^{n}(Q_i\chi_\rho(\alpha_i))^{\lambda_i}\prod_{j=1}^{m}\frac{1}{\lambda'_j!}\left(\frac{t_{j,\rho}\chi_\rho(\sum_{i=1}^{n}a_i(s'_j)\alpha_i)}{z}\right)^{\lambda'_j}y^{v',\rho}\times\\ &\times\frac{\prod_{\langle b\rangle=v_i,b\le 0}(y^{b'_i,\rho}+bz)}{\prod_{\langle b\rangle=v_i,b\le\lambda_i}(y^{b'_i,\rho}+bz)}\\ =&z\sum_{v'\in \mathrm{Box}(\mathfrak{X}(\mathbf{\Sigma}'))}\sum_{\lambda'\in\Lambda E^{S'}_{v'}}\prod_{i=1}^{n}(Q_i\chi_\rho(\alpha_i))^{\lambda_i}\prod_{j=1}^{m}\left(t_{j,\rho}\chi_\rho(\sum_{i=1}^{n}a_i(s'_j)\alpha_i)\right)^{\lambda'_j}y^{v',\rho}\times\\ &\times\frac{\prod_{\langle b\rangle=v_i,b\le 0}(y^{b'_i,\rho}+bz)}{\prod_{\langle b\rangle=v_i,b\le\lambda_i}(y^{b'_i,\rho}+bz)}\prod_{j=1}^{m}\frac{1}{\lambda'_j!z^{\lambda'_j}}.\end{aligned}$$

Note that this is the S'-extended I-function of $\mathfrak{X}(\mathbf{\Sigma}')$ under the change of variables,

$$e^{-\sum_{i=1}^{n}y^{b'_i,\rho}\log(Q_i\chi_\rho(\alpha_i))}I^{S'}_{\mathfrak{X}(\mathbf{\Sigma}')}(\{Q_i\chi_\rho(\alpha_i)\},\{t_{j,\rho}\chi_\rho(\sum_{i=1}^{n}a_i(s'_j)\alpha_i)\},z).$$

Again by Lemma 3.6, we have

$$\begin{aligned}&e^{\sum_{i=1}^{n}y^{b_i}\log Q_i/z}\cdot y^{v',\rho}=e^{\sum_{i=1}^{n}y^{b'_i,\rho}\log Q_i/z}\cdot y^{v',\rho}\\ =&e^{-\sum_{i=1}^{n}y^{b'_i,\rho}\log(\chi_\rho(\alpha_i))/z}e^{\sum_{i=1}^{n}y^{b'_i,\rho}\log(Q_i\chi_\rho(\alpha_i))/z}\cdot y^{v',\rho}.\end{aligned} \tag{41}$$

Thus we obtain

Theorem 4.2.

$$\begin{aligned}&I^{S}_{\mathfrak{X}(\mathbf{\Sigma})}(Q,t,z)\\ =&\sum_{\rho\in\widehat{A}}e^{-\sum_{i=1}^{n}y^{b'_i,\rho}\log(\chi_\rho(\alpha_i))/z}I^{S'}_{\mathfrak{X}(\mathbf{\Sigma}')}(\{Q_i\chi_\rho(\alpha_i)\},\{t_{j,\rho}\chi_\rho(\sum_{i=1}^{n}a_i(s'_j)\alpha_i)\},z).\end{aligned} \tag{42}$$

Now we take $S'\subset N'$ to contain the closed box $\overline{\mathrm{Box}}(\mathbf{\Sigma}')$. Since $\overline{\mathrm{Box}}(\mathbf{\Sigma})=\overline{\mathrm{Box}}(\mathbf{\Sigma}')\times A$, we see that the set $S\subset N$ contains $\overline{\mathrm{Box}}(\mathbf{\Sigma})$. By Theorem 4.1, this choice implies the following

(1) $t\mapsto I^{S}_{\mathfrak{X}(\mathbf{\Sigma})}(Q,t,z)$ is a family (of dimension $\ge H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}),\mathbb{C})$) of elements of the Lagrangian cone $\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma})}$ which is transverse to the ruling.

(2) $t \mapsto I^{S'}_{\mathfrak{X}(\mathbf{\Sigma}')}(Q,t,z)$ is a family (of dimension $\geq H^*_{CR}(\mathfrak{X}(\mathbf{\Sigma}'),\mathbb{C})$) of elements of the Lagrangian cone $\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma}')}$ which is transverse to the ruling.

The main point here is that, since these families have high enough dimensions, they can used to reconstruct the whole Lagrangian cones within the relevant symplectic vector spaces. Namely $t \mapsto I^{S}_{\mathfrak{X}(\mathbf{\Sigma})}(Q,t,z)$ reconstructs the Lagrangian cone $\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma})} \subset \mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma})}$, and the family $t \mapsto I^{S'}_{\mathfrak{X}(\mathbf{\Sigma}')}(\{Q_i\chi_\rho(\alpha_i)\},\{t_{j,\rho}\chi_\rho(\sum_{i=1}^n a_i(s'_j)\alpha_i)\},z)$ reconstructs the Lagrangian cone $\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma}'),\rho}|_{Q_i\mapsto Q_i\chi_\rho(\alpha_i)} \subset \mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma}'),\rho}|_{Q_i\mapsto Q_i\chi_\rho(\alpha_i)}$.

By divisor equation, the operator $e^{\sum_{i=1}^n y^{b'_i,\rho}\log(\chi_\rho(\alpha_i))/z}$ does the rescaling $Q_i \mapsto Q_i\chi_\rho(\alpha_i)$ for the ground ring $\Lambda_{nov}(\mathfrak{X}(\mathbf{\Sigma}'))$. In other words, $e^{\sum_{i=1}^n y^{b'_i,\rho}\log(\chi_\rho(\alpha_i))/z}$ identifies $\mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma}'),\rho}$ with $\mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma}'),\rho}|_{Q_i\mapsto Q_i\chi_\rho(\alpha_i)}$. Now Theorem 4.2 implies

Theorem 4.3. *The map (20) together with the rescaling $Q_i \mapsto Q_i\chi_\rho(\alpha_i), 1 \leq i \leq n$ identifies the direct sum of Lagrangian cones $\bigoplus_{[\rho]\in\widehat{A}}\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma}'),\rho}$ with the Lagrangian cone $\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma})}|_{Q_i\mapsto Q_i\chi_\rho(\alpha_i)}$.*

Recall that the J-function $J_{\mathfrak{X}}(t,-z)$ can be obtained as the intersection of the Lagrangian cone $\mathcal{L}_{\mathfrak{X}}$ and the affine space $-z+t+\mathcal{H}^{\mathfrak{X}}_+$. Thus it follows from Theorems 3.5 and 4.3 that

$$\begin{aligned} &J_{\mathfrak{X}(\mathbf{\Sigma})}(\{Q_i\},\{t_{j,\alpha}\},z)\\ =&\frac{1}{|A|^2}\sum_{\rho\in\widehat{A}} e^{-\sum_{i=1}^n y^{b'_i,\rho}\log(\chi_\rho(\alpha_i))/z} J_{\mathfrak{X}(\mathbf{\Sigma}')}(\{Q_i\chi_\rho(\alpha_i)\},\{t_{j,\rho}\},z). \end{aligned} \tag{43}$$

Here the factor $1/|A|^2$ accounts for the fact that we use the *scaled* orbifold Poincaré pairing $\frac{1}{|A|^2}(-,-)^{\mathfrak{X}(\mathbf{\Sigma}')}_{orb}$. In fact more is true. As explained in [13, Section 2], the Frobenius structure defined by the genus 0 Gromov-Witten theory of $\mathfrak{X}(\mathbf{\Sigma})$ is determined by the Lagrangian cone $\mathcal{L}_{\mathfrak{X}(\mathbf{\Sigma})}$ and the subspace $\mathcal{H}^{\mathfrak{X}(\mathbf{\Sigma})}_+$ (and similarly for the Frobenius structure defined by genus 0 Gromov-Witten theory of $\mathfrak{X}(\mathbf{\Sigma}')$). We thus deduce the following

Theorem 4.4. *The additive isomorphism (9) together with the identifications $Q_i \mapsto Q_i\chi_\rho(\alpha_i)$ give an isomorphism of Frobenius structures*

$$\begin{aligned} &\bigoplus_{[\rho]\in\widehat{A}}(QH^*_{orb}(\mathfrak{X}(\mathbf{\Sigma}),\Lambda_{nov}(\mathfrak{X}(\mathbf{\Sigma}'))),\frac{1}{|A|^2}(-,-)^{\mathfrak{X}(\mathbf{\Sigma}')}_{orb})\\ &\simeq (QH^*_{orb}(\mathfrak{X}(\mathbf{\Sigma}),\Lambda_{nov}(\mathfrak{X}(\mathbf{\Sigma}))),(-,-)^{\mathfrak{X}(\mathbf{\Sigma})}_{orb}). \end{aligned} \tag{44}$$

This in particular implies that the additive isomorphism (9) together with the identifications $Q_i \mapsto Q_i\chi_\rho(\alpha_i)$ give an isomorphism of quantum cohomology rings.

Example 4.5. We illustrate the aforementioned isomorphism of quantum cohomology rings in the example $\mathbb{P}(4,6)\to\mathbb{P}(2,3)$.

It is easy to check that $\mathbb{P}(4,6)\to\mathbb{P}(2,3)$ is the μ_2-gerbe obtained as the stack of square roots of $\mathcal{O}_{\mathbb{P}(2,3)}(1)$. In [2] the quantum cohomology rings of $\mathbb{P}(4,6)$ and $\mathbb{P}(2,3)$ (and more generally all weighted projective lines) are computed:

$$QH^*_{orb}(\mathbb{P}(2,3),\mathbb{C}) \simeq \mathbb{C}[[q]][x,y]/(xy-q,2x^2-3y^3),$$
$$QH^*_{orb}(\mathbb{P}(4,6),\mathbb{C}) \simeq \mathbb{C}[[q]][u,v,\xi]/(uv-q\xi,2u^2\xi-3v^3,\xi^2-1).$$

For $i = 0, 1$ let $QH^*_{orb}(\mathbb{P}(2,3), \mathbb{C})_i$ be a copy of $QH^*_{orb}(\mathbb{P}(2,3), \mathbb{C})$ with generators x_i, y_i and q rescaled by $(-1)^i$:

$$QH^*_{orb}(\mathbb{P}(2,3), \mathbb{C})_i = \mathbb{C}[[q]][x_i, y_i]/(x_i y_i - (-1)^i q, 2x_i^2 - 3y_i^3).$$

Let $\mathbf{1}_0 := \frac{1}{2}(1+\xi), \mathbf{1}_1 := \frac{1}{2}(1-\xi)$ and $u_i := (-1)^i u\mathbf{1}_i, v_i := (-1)^i v\mathbf{1}_i$. Then it is easy to check that the additive basis $\{\mathbf{1}_i, u_i, v_i, v_i^2 | i = 0, 1\}$ determines an isomorphism of algebras:

$$QH^*_{orb}(\mathbb{P}(4,6), \mathbb{C}) \simeq QH^*_{orb}(\mathbb{P}(2,3), \mathbb{C})_0 \oplus QH^*_{orb}(\mathbb{P}(2,3), \mathbb{C})_1,$$
$$\mathbf{1}_i \mapsto 1 \in QH^*_{orb}(\mathbb{P}(2,3), \mathbb{C})_i, u_i \mapsto x_i, v_i \mapsto y_i.$$

For instance,

$$\begin{aligned}
&\mathbf{1}_0\mathbf{1}_1 = \frac{1}{4}(1-\xi^2) = 0, \quad \mathbf{1}_0\mathbf{1}_0 = \frac{1}{2}(1+\xi) = \mathbf{1}_0, \quad \mathbf{1}_1\mathbf{1}_1 = \mathbf{1}_1,\\
&u_0 v_1 = 0, \quad u_1 v_0 = 0,\\
&u_0 v_0 = \frac{1}{2}(uv + uv\xi) = uv\mathbf{1}_0 = q\xi\mathbf{1}_0 = q\mathbf{1}_0,\\
&u_1 v_1 = \frac{1}{2}(uv - uv\xi) = uv\frac{1}{2}(1-\xi) = q\xi\frac{1}{2}(1-\xi) = q\frac{1}{2}(\xi - 1) = -q\mathbf{1}_1,\\
&2u_i^2 = 2u^2\mathbf{1}_i = 3v^3\xi\mathbf{1}_i = 3v^3(-1)^i\mathbf{1}_i = 3v_i^3, \quad i = 0, 1.
\end{aligned}$$

Remark 4.6. It is natural to ask for a generalization of our results to higher genus Gromov-Witten theory. One way to do this is as follows. First, By Theorem 4.4, we can identify the Frobenius structure of $\mathfrak{X}(\mathbf{\Sigma})$ with a direct sum of the Frobenius structure of $\mathfrak{X}(\mathbf{\Sigma}')$ (with a scaled pairing). By the results of [10], the Frobenius structure defined by the genus 0 Gromov-Witten theory of a toric Deligne-Mumford stack is generically *semi-simple*. By the theory of Givental [15], higher genus Gromov-Witten invariants of toric Deligne-Mumford stacks can be reconstructed from the Frobenius structures. We may then deduce results on higher genus *ancestor* Gromov-Witten invariants of a toric gerbe (1) by applying this reconstruction procedure to both sides of Theorem 4.4. We will not spell out the details here.

References

[1] D. Abramovich, T. Graber and A. Vistoli, Gromov–Witten theory of Deligne–Mumford stacks, *Amer. J. Math.* 130 (2008), no. 5, 1337–1398, math.AG/0603151.

[2] D. Abramovich, T. Graber and A. Vistoli, Algebraic orbifold quantum product, in *Orbifolds in mathematics and physics (Madison,WI,2001)*, 1–24, *Contem. Math.* 310, Amer. Math. Soc., 2002, math.AG/0112004.

[3] E. Andreini, Y. Jiang and H.-H. Tseng, Gromov-Witten theory of product stacks, *Comm. Anal. Geom.* 24 (2016), no. 2, 223–277, arXiv:0905.2258.

[4] E. Andreini, Y. Jiang and H.-H. Tseng, Gromov-Witten theory of root gerbes, I: structure of genus 0 moduli spaces, *J. Differential Geom.* 99 (2015), no. 1, 1–45, arXiv:0907.2087.

[5] E. Andreini, Y. Jiang and H.-H. Tseng, Gromov-Witten theory of banded gerbes over schemes, arXiv:1101.5996.

[6] L. Borisov, L. Chen and G. Smith, The orbifold Chow ring of toric Deligne-Mumford stacks, *J. Amer. Math. Soc.* 18 (2005), no.1, 193–215.

[7] W. Chen and Y. Ruan, A new cohomology theory for orbifolds, *Comm. Math. Phys.* 248 (2004), no. 1, 1–31.

[8] W. Chen and Y. Ruan, Orbifold Gromov-Witten theory, in *Orbifolds in mathematics and physics (Madison, WI, 2001)*, 25–85, *Contem. Math.* 310, Amer. Math. Soc., 2002. math.AG/0103156.

[9] T. Coates, A. Corti, H. Iritari and H.-H. Tseng, Computing genus-zero twisted Gromov-Witten invariants, *Duke Math. J.*, 147, No. 3 (2009), 377–438, arXiv:math/0702234.

[10] T. Coates, A. Corti, H. Iritari and H.-H. Tseng, A mirror theorem for toric stacks, *Comp. Math.* 151 (2015) 1878–1912.

[11] T. Coates, A. Corti, Y.-P. Lee and H.-H. Tseng, The quantum orbifold cohomology of weighted projective spaces, *Acta Math.*, 202, No. 2 (2009), 139–193, arXiv:math/0608481.

[12] T. Coates, and A. Givental, Quantum Riemann-Roch, Lefschetz and Serre, *Ann. of Math.* (2) 165 (2007), no. 1, 15–53.

[13] T. Coates, H. Iritari and H.-H. Tseng, Wall-Crossings in Toric Gromov-Witten Theory I: Crepant Examples, *Geom. Topol.* 13 (2009), 2675–2744, arXiv:math.AG/0611550.

[14] B. Fantechi, E. Mann and P. Nironi, Smooth toric DM stacks, *J. Reine Angew. Math.* 648 (2010), 201–244, arXiv:0708.1254.

[15] A. Givental, Gromov-Witten invariants and quantization of quadratic Hamiltonians, *Mosc. Math. J.* 1(2001), no. 4, 551–568.

[16] A. Givental, Symplectic geometry of Frobenius structures, *Frobenius manifolds*, 91–112, Aspects Math., E36, Vieweg, Wiesbaden, 2004, math.AG/0305409.

[17] J. Giraud, Cohomologie non abélienne, Springer-Verlag Berlin 1971.

[18] S. Hellerman, A. Henriques, Tony Pantev and Eric Sharpe, Cluster decomposition, *T*-duality, and gerby CFT's, *Adv. Theor. Math. Phys.* 11 (2007), no. 5, 751–818, arXiv:hep-th/0606034.

[19] Y. Jiang, The orbifold cohomology of simplicial toric stack bundles, *Illinois J. Math.*, 52, No.2 (2008), 493–514, math.AG/0504563.

[20] Y. Jiang and H.-H. Tseng, The integral (orbifold) Chow ring of toric Deligne-Mumford stacks, *Math. Z.*, 264, No. 1 (2010), 225–248, arXiv:0707:2972.

[21] P. Johnson, Equivariant Gromov-Witten theory of one dimensional stacks, *Comm. Math. Phys.* (2014), 327, no. 2, 333–386.

[22] X. Tang and H.-H. Tseng, Duality theorems of étale gerbes on orbifolds, *Adv. Math.* 250 (2014), 496–569, arXiv:1004.1376.

[23] X. Tang and H.-H. Tseng, A quantum Leray-Hirsch theorem for banded gerbes, arXiv:1602.03564.

[24] H.-H. Tseng, Orbifold Quantum Riemann-Roch, Lefschetz and Serre, *Geom. Topol.* 14 (2010) 1–81.