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THE SYLOW STRUCTURE OF TRANSITIVE PERMUTATION GROUPS ACTING WITH FIXITY 4

BARBARA BAUMEISTER, KAY MAGAARD, AND REBECCA WALDECKER

Dedicated to the memory of Kay Magaard.

ABSTRACT. We study finite permutation groups with special properties, motivated from the theory of Riemann surfaces. In this article we focus on groups acting with fixity 4 and analyse their Sylow structure and possible orbit lengths of Sylow subgroups.

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KEYWORDS: Sylow structure, transitive permutation group

1. INTRODUCTION

The nature of the fixed point sets of group actions continues to play a central role in group theory. This paper continues our investigation of transitive permutation groups in which all nonidentity elements fix at most four points, with a focus on the Sylow structure of such groups. We decided to summarize basic results about orbit sizes of Sylow subgroups in this article, in particular some information for the primes 2 and 3, because this information is much more intricate and complex than it was in previous work on groups acting with fixity 2 or 3.

Building on this work, subsequent papers will give more details of the Sylow structure analysis and consequences for the group structure in general, along with a classification of finite simple groups acting with fixity 4. While our class of permutation groups is of purely group theoretic interest, our initial interest in this topic stems from the study of Weierstraß points of Riemann surfaces, and we refer the reader to [6] for more background.

Acknowledgments. The project on permutation groups with low fixity started with support by the DFG. It was initiated by Kay Magaard and most of the work in this paper has been completed during his visits at the Martin-Luther-Universität Halle-Wittenberg in 2016 and 2017. Work on the project has continued after his passing in July 2018 and will continue until his initial goal has been reached.

2. Notation and examples

Let $k \in \mathbb{N}_0$ and suppose that the group G acts on the finite set Ω. We say that G has fixity k on Ω if and only if there is some element of G that fixes exactly k distinct points on Ω and if no element of G fixes more than k distinct points.

Hypothesis 1. *Suppose that (G,Ω) is such that G acts faithfully and transitively on Ω and that G has fixity 4 in this action.*

We begin with natural and well-known examples.

Example 1 (Classical examples)**.** *Suppose that the group G acts sharply 5-transitively on a set Ω of size at least 6. Then for all pair-wise distinct elements $\omega_1,\omega_2,\omega_3,\omega_4 \in \Omega$, we find α,β in $\Omega \setminus \{\omega_1,\omega_2,\omega_3,\omega_4\}$ that are distinct, and then the 5-transitivity gives an element $g \in G$ such that $\omega_1,\omega_2,\omega_3,\omega_4$ are fixed and α is mapped to β. In particular every four point stabilizer is non-trivial, but every five point stabilizer is trivial because of the sharp 5-transitivity. Therefore (G,Ω) satisfies Hypothesis 1.*

Now Jordan's result (see for example p. 327 in [5]) this gives the well-known examples $\mathcal{S}_5$, $\mathcal{S}_6$, $\mathcal{A}_7$, M_{12}, in their natural action. The transitive subgroup $M_{11} \leq M_{12}$ also gives rise to an example, with point stabilizers of order $2^2 \cdot 3 \cdot 5 \cdot 11$.

Calculations in GAP ([8]) show that the groups $\mathcal{S}_5$, $\mathcal{A}_6$ and $\mathcal{S}_6$ also give rise to several (very small) examples, respectively.

Example 2 (Frobenius group extensions)**.** *Suppose that F is a Frobenius group with Frobenius kernel K and complement H and suppose that G is a group with a subgroup U of order 4 such that $G = F \rtimes U$.*

Consider the action of G on G/H by right multiplication. As all conjugates of H are contained in F, it follows that $|N_G(H)| = 4 \cdot |H|$. If $g \in G$ is such that $g \in N_G(H) \setminus H$, then all elements of H fix the coset Hg. This gives four fixed points on the set G/H, and it follows that $(G, G/H)$ satisfies Hypothesis 1 in the special case where all elements have 0 or 4 fixed points.

A similar construction works based on pairs (G,Ω) that satisfy the main hypothesis of [6].

Example 3. *Suppose that G is a finite group, that $U \leq G$ has order 2 and that $H, M \leq G$ are such that the following hold: $H \leq M$, $G = M \rtimes U$ and with respect to the action by right multiplication, the group M has fixity 2 on M/H. In particular $|N_M(H) : H| = 2$. As M has index 2 in G, all conjugates of H in G are contained in M and $|N_G(H) : H| = 4$. It follows that $(G, G/H)$ satisfies Hypothesis 1 with the action of G on G/H by right multiplication.*

3. General properties

By "group" we always mean a finite group, and by "permutation group" we always mean a group that acts faithfully. Throughout this paper Ω denotes a finite set and G denotes a permutation group on Ω. We also use the notion of fixity introduced in the previous section.

Let $\omega \in \Omega$ and $g \in G$, and moreover let $\Lambda \subseteq \Omega$ and $H \leq G$. Then $H_\omega := \{h \in H \mid \omega^h = \omega\}$ denotes the **stabilizer of** ω **in** H,

$$\text{fix}_\Lambda(H) := \{\omega \in \Lambda \mid \omega^h = \omega\}$$

for all $h \in H\}$ denotes the **fixed point set of** H **in** Λ and we write $\mathrm{fix}_\Lambda(g)$ instead of $\mathrm{fix}_\Lambda(\langle g\rangle)$. We write ω^H for the H-orbit in Ω that contains ω.

Whenever $n, m \in \mathbb{N}$ and p is a prime number, then we denote by (n, m) the largest natural common divisor of n and m and by n_p the largest power of p dividing n.

Finally, a subgroup H of G is said to be a **TI-subgroup** if and only if for all $g \in G$, we have that $H^g \cap H = 1$ or $H^g = H$.

We begin with general local properties that follow from our main hypothesis. We will notice that the primes 2 and 3 behave differently from the larger primes in $\pi(G)$, motivating the extensive analysis that we begin to describe in this article.

Lemma 4. *Suppose that Hypothesis 1 holds and let $\alpha \in \Omega$. If $p \in \pi(G_\alpha)$ and $p \geq 5$, then G_α contains a Sylow p-subgroup of G.*

If moreover some non-trivial p-element fixes four points, then the corresponding four point stabilizer contains a Sylow p-subgroup of G.

Proof. Let $Q \in \mathrm{Syl}_p(G_\alpha)$ and let $Q \leq P \in \mathrm{Syl}_p(G)$. Then $Z(P)$ centralises Q and therefore stabilizes the set $\mathrm{fix}_\Omega(Q)$ of size at most 4, by hypothesis. In particular $Z(P)$ induces a subgroup of $\mathcal{S}_4$ on this set, but it has order divisible by $p \geq 5$. This implies that $Z(P)$ fixes every element of $\mathrm{fix}_\Omega(Q)$, in particular $Z(P) \leq G_\alpha$. With the same argument P fixes every fixed point of $Z(P)$, including α, and therefore $P \leq G_\alpha$.

Now let $1 \neq x \in P$ be such that x fixes four points. Let $\Delta := \mathrm{fix}_\Omega(x)$. Then we argue as above: First $Z(P)$ centralizes x and hence fixes Δ point-wise, and then P also stabilizes Δ point-wise. □

From this we obtain initial information about $F^*(G)$:

Corollary 5. *Suppose that Hypothesis 1 holds, let $\alpha \in \Omega$ and let $p \in \pi(G)$. If $P \in Syl_p(G)$ and $P \leq G_\alpha$, then $O_p(G) = 1$. In particular, if $p > 3$ and $p \in \pi(G_\alpha)$, then $O_p(G) = 1$.*

Proof. The first statement follows from the fact that G acts faithfully and transitively on Ω. Then the second claim follows form the first, together with Lemma 4. □

Lemma 6. *Suppose that Hypothesis 1 holds. Let $\alpha \in \Omega$ and suppose that $H \leq G_\alpha$ is a non-trivial 4-point stabilizer.*

(a) H is a TI-subgroup.
(b) If 3 divides $\pi(H)$, then H contains a Sylow 3-subgroup of G or $N_G(H)$ has a subgroup that induces A_4 on $\mathrm{fix}_\Omega(H)$.
(c) If $N_G(H)$ is not transitive on $\mathrm{fix}_\Omega(H)$, then $N_G(H)/H$ is elementary abelian of order 4 and $N_{G_\alpha}(H)/H$ of order 2.
(d) If $1 \neq X \leq G_\alpha$ is a subgroup that fixes exactly one or two points, then $|N_G(X) : N_{G_\alpha}(X)| \in \{1, 2\}$. If X fixes exactly four points, then $N_G(X) \leq N_G(H)$ and $|N_G(H) : N_{G_\alpha}(H)| \in \{2, 4\}$. In particular, if $3 \in \pi(N_G(X))$ in this case, then $3 \in \pi(G_\alpha)$.

Proof. Set $\Delta := \mathrm{fix}_\Omega(H)$ and let $g \in G$. Then $H \cap H^g$ fixes Δ and Δ^g point-wise, so $\mathrm{fix}_\Omega(H \cap H^g)$ contains Δ and Δ^g. If $H \cap H^g \neq 1$, then Hypothesis 1 forces $|\Delta \cup \Delta^g| \leq 4$ and therefore $\Delta = \Delta^g$ and $H = H^g$. This is (a).

The hypothesis in (b) implies that some nontrivial 3-element fixes four points, hence $|\Omega| \equiv 1 \mod 3$. Therefore G_α contains a Sylow 3-subgroup of G. Suppose that H does not contain a Sylow 3-subgroup of G. Then there exists a 3-element in G_α that induces a 3-cycle on Δ. Such an element stabilizes the set Δ and therefore normalizes H. We conclude that 3 divides $|N_{G_\alpha}(H)/H|$. If $\beta \in \Delta$ and $\beta \neq \alpha$, then $H \leq G_\beta$ and we can argue in the same way to see that 3 divides $|N_{G_\beta}(H)/H|$. In particular we find two elements that induce distinct 3-cycles on Δ, hence there is a subgroup of $N_G(H)$ that induces $\mathcal{A}_4$ on Δ. Thus (b) holds.

For (c) we suppose that $N_G(H)$ is not transitive on Δ, which means that $|N_G(H) : N_{G_\alpha}(H)| \in \{1, 2, 3\}$.

We show that H is a 2-group. As a first step we prove

$(*)$ For all $r \in \pi(G)$, H does not contain a Sylow r-subgroup of G_α.

Assume otherwise and let $r \in \pi(G)$, $R \in \mathrm{Syl}_r(G_\alpha)$ and $R \leq H$. Let $\beta \in \Delta$ be such that $\beta \neq \alpha$. In particular $R \leq H \leq G_\beta$ and then $R \in \mathrm{Syl}_r(G_\beta)$. As G is transitive on Ω, there is some $g \in G$ such that $\alpha^g = \beta$, and then R and R^g are Sylow r-subgroups of G_β. By Sylow's Theorem let $x \in G_\beta$ be such that $R^{gx} = R$. Then gx normalizes R and hence stabilizes the set $\mathrm{fix}_\Omega(R) = \Delta$, in particular $gx \in N_G(H)$. As $\alpha^{gx} = \beta$, we now have the contradiction that $N_G(H)$ acts transitively on Δ. This proves $(*)$.

If $p \in \pi(H)$ and $p \geq 5$, then Lemma 4 yields that H contains a Sylow p-subgroup P of G, contrary to $(*)$.

If $3 \in \pi(H)$, then we recall that H des not contain a Sylow 3-subgroup of G_α (by $(*)$ and therefore a subgroup of $N_G(H)$ induces $\mathcal{A}_4$ on Δ by (b). But this contradicts our hypothesis that $N_G(H)$ is not transitive on Δ.

We conclude that H is a 2-group.

Now $(*)$ implies that H is not a Sylow 2-subgroup of G_α. Let $H \leq S \in \mathrm{Syl}_2(G_\alpha)$ and $T := N_S(H)$. Then $H < T$ and therefore some $t \in T$ induces a transposition on Δ. It follows for all $\delta \in \Delta$ that some element of $N_{G_\delta}(H)$ induces a transposition on Δ. Together with the fact that $N_G(H)/H$ is isomorphic to a non-transitive subgroup of $\mathcal{S}_4$, by hypothesis, we deduce that (c) holds.

For (d) we let $\Lambda := \mathrm{fix}_\Omega(X)$. If $|\Lambda| \leq 2$, then the first assertion of (d) holds.

Now suppose that $|\Lambda| = 4$ and let M denote the point-wise stabilizer of Λ. As $N_G(X)$ stabilizes Λ, it normalizes M. If 3 is not in $\pi(N_G(M)/M))$, then (d) follows. So we suppose that $3 \in \pi(N_G(H)/H)$. Then $N_G(M)$ does not induce a 2-group on Γ and therefore $N_G(M)$ is transitive on Γ by (c), which shows the second assertion of (d).

Now suppose that $3 \in \pi(N_G(X))$. Then $3 \in \pi(N_G(M))$ and $3 \in \pi(M)$ or $3 \in \pi(N_G(M)/M))$ and $N_G(M)$ is transitive on Γ by the last paragraph. This shows the last assertion of (d).

□

Corollary 7. *Suppose that Hypothesis 1 holds. Then* $|Z(G)| \in \{1, 2, 4\}$.

Proof. Let $\alpha \in \Omega$. As G acts faithfully on Ω, we know that $Z := Z(G)$ intersects G_α trivially. Let $x \in G_\alpha$ be an element with exactly four fixed points. Then $Z \leq C_G(x)$ and hence Lemma 6 (d) implies our assertion. □

Lemma 8. *Suppose that Hypothesis 1 holds and let* $\alpha \in \Omega$. *Then the following hold:*

(a) If there is a 2-element in G_α *that has exactly one or three fixed points on* Ω*, then* G_α *contains a Sylow 2-subgroup of* G.

(b) If there is a 3-element in G_α that has exactly one, two or four fixed points, then G_α contains a Sylow 3-subgroup of G.

Proof. For (a) we suppose that $x \in G_\alpha$ is a 2-element with exactly one or three fixed points. As x has orbits of 2-power lengths on the set of points that are not fixed, it follows that Ω is odd. Therefore $|G : G_\alpha|$ is odd and consequently G_α contains a Sylow 2-subgroup of G.

Next suppose that $y \in G_\alpha$ is a 3-element with exactly one, two or four fixed points on Ω. We note that this implies that $|\Omega| \equiv 1$ or 2 mod 3 and hence G_α contains a Sylow 3-subgroup of G. This is (b). □

4. Orbit lengths for Sylow subgroups

We prove two basic lemmas that allow us to determine the possible orbit sizes for Sylow subgroups. After that we analyze the situation for the prime 2 in more detail.

Lemma 9. *Suppose that Hypothesis 1 holds. Let $S \in Syl_2(G)$ and $\alpha \in \Omega$. Then one of the following holds:*

(a) $S_\alpha = 1$.
(b) $|S_\alpha| = 2$ and S is dihedral or semidihedral.
(c) $1 \neq |S_\alpha| \leq 8$, $|\alpha^S| \geq 8$ and there exists some subgroup $T \leq S_\alpha$ of index at most 2 such that all $t \in T^\#$ fix exactly four points.
Moreover S_α is isomorphic to a subgroup of D_8.
(d) $|S : S_\alpha| \in \{2, 4\}$.
(e) $S \leq G_\alpha$.

Proof. We suppose that neither (a) nor (e) holds. Then $1 \neq S_\alpha \neq S$ and therefore the orbit $\Delta := \alpha^S$ is nonregular of length at least 2. If $|\Delta| \leq 4$, then (d) holds because $|\Delta| = |S : S_\alpha|$. So now we suppose that (d) does not hold.

Then $|\Delta| \geq 8$ and we consider (S, Δ). If this pair does not satisfy Hypothesis 1, then it satisfies Hypothesis 1.1 from [6]. Then it follows from Lemma 2.12 in this article that S is dihedral or semidihedral, which means that (b) holds.

So now we suppose that (S, Δ) satisfies Hypothesis 1 and we let $t \in S^\#$ be such that t fixes exactly four points on Δ. Without loss $\alpha \in \text{fix}_\Delta(t)$. Next we let T denote the point-wise stabilizer of $\text{fix}_\Delta(t)$ in S. Then $T \leq S_\alpha$ and, since T fixes four points, all other orbits of T on Δ are regular by Hypothesis 1.

We let $a \in \mathbb{N}_0$ be such that $|\Delta| = 4 \cdot a + |T|$. As $|\Delta|$ is a power of 2, at least 8, it follows that $|\Delta|$ is divisible by 8 and hence $a \cdot |T|$ is divisible by 4, but not by 8. This forces $|T| \leq 4$. Moreover $T \leq S_\alpha \neq S$ and hence $T < N_S(T)$. The factor group $N_S(T)/T$ is isomorphic to a 2-subgroup of $\mathcal{S}_4$ (i.e. a subgroup of D_8), and $|S_\alpha : T| \leq 2$. In particular $|S_\alpha| \leq 2 \cdot |T| \leq 8$, as in (c), and it only remains to prove that S_α is isomorphic to a subgroup of D_8.

This is clear if $|S_\alpha| \leq 4$. Otherwise $|S_\alpha| = 8$ and $|T| = 4$. If we recall the equation $|\Delta| = 4 \cdot a + |T|$ from above, then we obtain that a is odd. Let $s \in S_\alpha \setminus T$. Then s stabilizes at least one of the regular T-orbits on Δ, and on such an orbit it fixes at most two points. As every element of $T^\#$ acts as a 4-cycle or a double transposition on each regular T-orbit, it follows that $S_\alpha = \langle s, T \rangle \simeq D_8$. □

Lemma 10. *Suppose that Hypothesis 1 holds and let $P \in Syl_3(G)$. Let Δ be the union of all P-orbits of Ω of size at most 3. Then one of the following holds:*

(a) All P-orbits are regular and the point stabilizers in G are $3'$-groups.
(b) $|\Delta| > 4$ and $|P| \leq 9$.
(c) $|\Delta| \leq 4$ and P is of maximal class. There exists some nonregular P-orbit on $\Omega \setminus \Delta$ and for every such orbit Λ and all $\lambda \in \Lambda$ it is true that $|P_\lambda| = 3$ and that P_λ fixes exactly three points on Λ.
(d) Δ is the unique P-orbit of length 3 and all orbits of P on $\Omega \setminus \Delta$ are regular.
(e) $1 \leq |\Delta| \leq 4$, there is some $\delta \in \Delta$ such that $P \leq G_\delta$, and all P-orbits on $\Omega \setminus \Delta$ are regular.

In (c), (d) and (e) we see that P possesses an orbit of length at least 9 and therefore $|P| \geq 9$.

Proof. Suppose that $|\Delta| > 4$. As P is a 3-group, all P-orbits contained in Δ are of length 1 or 3. If $\Sigma_1, \Sigma_2 \subseteq \Delta$ are distinct P-orbits of length 3, then $|\Sigma_1 \cup \Sigma_2| = 6$ and therefore Hypothesis 1 implies that P acts faithfully on $\Sigma_1 \cup \Sigma_2$. In particular P is isomorphic to a subgroup of a Sylow 3-subgroup of $\mathcal{S}_6$ and hence $|P| \leq 9$. This is (b).

If there is a unique P-orbit $\Sigma_1 \subseteq \Sigma$ of length 3, then the other P-orbits contained in Δ have length one. Consequently the kernel P_0 of the action of P on Σ_1 fixes Δ point-wise. As $|\Delta| > 4$, it follows from Hypothesis 1 that P acts faithfully on Σ_1 and therefore $|P| \leq 3$. Again this is included in (b).

Now we suppose that neither (a) nor (b) holds, and from the previous paragraphs we know that this implies that $0 \leq |\Delta| \leq 4$.

Suppose further that P acts semiregularly on $\Omega \setminus \Delta$. In particular (c) does not hold. Assume that $|\Delta| = 0$, i.e. $\Delta = \varnothing$. Then all P-orbits are regular, which implies (a) and hence a contradiction. So we know that $|\Delta| \geq 1$ and we prove that (d) or (e) holds:

If P is contained in a point stabilizer, then the size of Δ can be anything between 1 and 4 because P can fix up to four points or P has one fixed point and one orbit of length 3. These cases are covered by (e).

Otherwise all P-orbits have length at least 3. Together with our restriction $0 \leq |\Delta| \leq 4$ this implies (d).

If none of (a),(b),(d) or (e) holds, then the previous arguments show that $0 \leq |\Delta| \leq 4$ and that P has a nonregular orbit on $\Omega \setminus \Delta$. Let Λ denote such a nonregular orbit. Then Λ has size at least 9 by definition of Δ.

Let $\lambda \in \Lambda$ and let $n \in \mathbb{N}$ be such that $|\Lambda| = 3^n$. Then $n \geq 2$ and by our main hypothesis P_λ fixes at most four points in total. So it follows that P_λ fixes exactly three points on Λ and acts semi-regularly on the set of remaining points. Let $m \in \mathbb{N}$ be such that $|P_\lambda| = 3^m$ and let $a \in \mathbb{N}$ be such that $|\Lambda| = 3 + a \cdot |P_\lambda|$. Then $3^n = |\Lambda| = 3 + a \cdot |P_\lambda| = 3 + a \cdot 3^m$, so this forces $m = 1$. We deduce that $|P_\lambda| = 3$ and now Lemma 6 implies that $|N_P(P_\lambda)| \leq 9$.

This means that P has maximal class and that P_λ fixes exactly three points in Λ, as stated in (c).

Finally we argue why the concluding remark is true. In (c), with the notation introduced there, we see that $|P_\lambda| = 3$ and $|\Lambda| \geq 3$, because P_λ fixes three points on Λ. This means that $|P| = |P : P_\lambda| \cdot |P_\lambda| \geq 3 \cdot 3 = 9$.

In (d) the regular orbits of P have size at least 3 because Δ is the unique orbit of size 3. As $|\Omega| > 3$, there must be a regular P-orbit and hence $|P| \geq 9$ again.

In (e) we see that $|\Delta| \leq 4$ and $|\Omega| \geq 6$, therefore $\Omega \setminus \Delta \neq \varnothing$ and regular orbits have size strictly greater than 3. Consequently $|P| \geq 9$. □

Lemma 11. *Suppose that Hypothesis 1 holds, let $p \in \pi(G)$ and let $P \in Syl_p(G)$. Then the following hold:*

(a) *If $p = 2$, then the possible orbit sizes for P on Ω are 1, 2, 4, 8, $\frac{|P|}{8}$, $\frac{|P|}{4}$, $\frac{|P|}{2}$ and $|P|$.*
(b) *If $p = 3$, then the possible orbit sizes for P on Ω are 1, 3, $\frac{|P|}{3}$ and $|P|$.*
(c) *If $p \geq 5$, then the possible orbit sizes for P on Ω are 1 and $|P|$.*

Proof. (a) follows from Lemma 9 by inspecting the cases.

For (b) we look at Lemma 10: In addition to orbits of size $|P|$ and 1 we see in (d) that orbit size 3 is possible, and the only case where another orbit size occurs is (c), with orbits of length $|P|/3$.

Finally, if $p \geq 5$, then we turn to Lemma 4. If P does not fix any point, then the lemma forces the point stabilizer orders to be coprime to p. This means that all P-orbits are regular. □

We remark that the group M_{12} in its natural action on 12 points provides an example for Case (b) where a 3-Sylow subgroup has one orbit of length 3 and another nonregular orbit.

For more details about the 2-structure we set up additional notation.

Hypothesis 2. *Suppose that Hypothesis 1 holds and let $S \in Syl_2(G)$. Let f denote the maximal number of points of Ω that are fixed by some involution in G, let Δ denote the union of S-orbits on Ω of length at most 4 and let*

$$F := \langle x^g \mid x, g \in G, o(x) = 2, |\, fix_\Omega(x)| = f\rangle.$$

We note that Hypothesis 2 allows for the special cases that $S = 1$ or $f = 0$.

Lemma 12. *Suppose that Hypothesis 2 holds and that the point stabilizers in G have even order. Then one of the following is true:*

(a) $|\Delta| \leq 4$, S acts semi-regularly on $\Omega \setminus \Delta$ and the stabilizer of the set Δ is strongly embedded in F.
(b) $|\Delta| > 4$ and S acts semi-regularly on $\Omega \setminus \Delta$. Moreover there exists a subset $\Delta_1 \subset$ of Δ that is S-invariant and such that $4 < |\Delta_1| \leq 8$.
(c) S does not act semi-regularly on $\Omega \setminus \Delta$ and there exists an S-orbit Λ in $\Omega \setminus \Delta$ such that $max_{s \in S^\#}\{|fix_\Lambda(s)|\} \in \{2, 4\}$.

Proof. Since G acts transitively on Ω and the maximum number of fixed points of an involution of G is 4, we see that the hypothesis of the main theorem and Proposition 3.1 of [7] are satisfied. Applied to $f = 4$, this gives exactly the statements in the three cases; it is worth mentioning that the value of k in Ronse's proposition 3.1 (ii) can only be 2 in our situation. □

Keeping the notation from Hypothesis 2 Ronse (see [7], main theorem) proves a more general result:

Theorem 13. *If G has even order and acts transitively on a set Ω such that $f \leq 4$, then one of the following holds:*

(a) The set stabilizer of Δ in F is strongly embedded in F, or
(b) $f \leq 3$ and the Sylow 2-subgroups are dihedral or semidihedral, or
(c) $f = 4$ and the Sylow 2-subgroups of G have sectional rank bounded by 4.

For later applications we need a version for simple groups.

Theorem 14. *Suppose that Hypothesis 2 holds and that G is nonabelian simple. Then one of the following is true:*

(a) The point stabilizers in G have odd order.
(b) G has a strongly embedded subgroup and is therefore isomorphic to $PSL_2(q)$, $Sz(q)$ or $PSU_3(q)$ for s suitable power q of the prime 2, where $q \geq 4$.
(c) $f \leq 3$ and G is isomorphic to $\mathcal{A}_7$, to M_{11} or to $PSL_2(q)$ for some prime power q or to $PSL_3(q)$ or $PSU_3(q)$ for some odd number q that is a prime power.
(d) $f = 4$ and G has sectional 2-rank at most 4.

Proof. Suppose that the point stabilizers have even order. Then Theorem 13 applies and $F = G$ because G is simple. So in Case (a) of the theorem, we see that G has a strongly embedded subgroup and Bender's classification gives our statement in (b). (See [2].) Case (b) of the theorem leads to dihedral or semidihedral Sylow 2-subgroups and hence to the classification results by [4] and [1], giving our list of groups in (c). Finally Case (c) in Theorem 13 directly gives our claim (d). □

If the point stabilizers have odd order, then it becomes important whether or not 3 divides their order. This connection between the primes 2 and 3 will be discussed in subsequent work. In Case (d) above, the group G is known by work of Gorenstein and Harada (see [3]).

5. Final comments

As we have discussed in the introduction, analyzing the Sylow structure of groups satisfying Hypothesis 1 is crucial when it comes to classifying these groups.

Corollary 5 shows where the differences between the primes 2 and 3 an larger primes come into play, and with its help we can prove the following:

If Hypothesis 1 holds and $\omega \in \Omega$, then all Sylow subgroups of G_ω have rank 1 or

$$F^*(G) = O_2(G)O_3(G)E(G).$$

This leads to a natural case distinction not only for primes that divide the order of point stabilizers, but also for the rank of their Sylow subgroups. Following this case distinction we will classify all finite simple groups (and some extensions) satisfying Hypothesis 1, and we are also working on general structure results.

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FAKULTÄT FÜR MATHEMATIK,, UNIVERSITÄT BIELEFELD, POSTFACH 10 01 31,, 33501 BIELEFELD, GERMANY.

E-mail address: B.Baumeister@math.uni-bielefeld.de

INSTITUT FÜR MATHEMATIK, MLU HALLE-WITTENBERG, THEODOR-LIESER-STRASSE 5, 06120 HALLE, GERMANY.

E-mail address: rebecca.waldecker@mathematik.uni-halle.de