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USING STRONG BRANCHING TO FIND AUTOMORPHISM GROUPS OF n-GONAL SURFACES

S. ALLEN BROUGHTON, CHARLES CAMACHO, JENNIFER PAULHUS, REBECCA R. WINARSKI, AND AARON WOOTTON

Dedicated to the memory of Kay Magaard

Abstract. The problem of finding full automorphism groups of compact Riemann surfaces is classical, though complete results are only known for a few families. One tool used in some classification schemes is strong branching; a condition derived by Accola in [1]. In the following, we survey the main ideas behind strong branching including a general survey of current results. We also provide new results for families for which we can find the full automorphism group using strong branching and an inductive version of strong branching.

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1. Introduction

Ideally, we would like to be able to determine the full automorphism group of a Riemann surface given some partial information about the surface such as defining equations, uniformization by a Fuchsian group, a branched covering map to a known surface, or a "sufficiently large"group of automorphisms. In this paper we are particularly interested in the interplay of the last two items. For our purposes, a subgroup G of the automorphism group of a Riemann surface S is called "sufficiently large" if S/G has genus zero. Alternatively, S is called a *regular n-gonal surface*. A regular n-gonal surface is one for which the quotient map $\pi_G : S \to S/G \simeq \mathbb{P}^1(\mathbb{C})$ is a regular branched covering of the sphere $\mathbb{P}^1(\mathbb{C})$ of degree $n = |G|$, branched over a finite set of points $B_G = \{Q_1, \ldots, Q_t\}$. This class of surfaces includes these important cases: hyperelliptic surfaces, superelliptic surfaces, cyclic n-gonal surfaces, quasi-platonic surfaces, as well as many others. In the moduli space of surfaces of fixed genus $\sigma \geq 2$ the "most common" surfaces with automorphisms are regular n-gonal surfaces. We use this fact and the important cases described above as justification for focusing on the study on regular n-gonal surfaces. The notion of "most common" can be made precise using Breuer's data on low genus actions [5], see the end of Section 4.2.

Finding automorphism groups of n-gonal surfaces.

Let us describe an approach to finding the full automorphism group of an n-gonal surface. We assume we are given a group, G, of automorphisms with genus zero quotient S/G. We will also assume that we have very precise information about how G acts on S and the map $\pi_G : S \to \mathbb{P}^1(\mathbb{C})$. Throughout the paper let $A = \mathrm{Aut}(S)$, $N = \mathrm{Nor}_A(G)$, and $K = N/G$. Since N normalizes G, then K acts as a group of automorphisms of $S/G \simeq \mathbb{P}^1(\mathbb{C})$. The candidate groups K are precisely known and given the structure of the map $\pi_G : S \to \mathbb{P}^1(\mathbb{C})$, the structure of the group N may be determined, as well as the map $\pi_N : S \to S/N \simeq \mathbb{P}^1(\mathbb{C})$. See Section 3.1 for details.

If $N = A$ then we are done. Otherwise we have *exceptional automorphisms* in $A - N$. The branched covering $\pi_{A/N} : S/N \to S/A$ is a rational map of the sphere, and its monodromy can be determined. The monodromy may then be used to construct the extension $N < A$. The latter situation is unusual and takes considerable work using MAGMA [4] or GAP [11] to solve. See Section 3.2 for details.

A tricky step in the aforementioned process is deciding whether or not G is normal in A without any prior knowledge of A. In general, an answer to this question is likely very difficult. However, if the map π_G is *strongly branched* – a concept introduced by Accola [1] – G is guaranteed to have a subgroup M that is normal in A. Strong branching is checked by an easily verifiable inequality. Using strong branching, the classification process splits up into two cases:

(1) The genus of S is larger than a lower bound determined by strong branching, and there is a normal subgroup $M \trianglelefteq A$ contained in G. If $G = M$, then $A = N$, and we can compute A as described above. If M is a proper subgroup of G then $\overline{S} = S/M$ is a surface upon which both $\overline{A} = A/M$ and $\overline{G} = G/M$ act, and $\overline{A} \leq \mathrm{Aut}(\overline{S})$. Presumably we can compute $\overline{A} \leq \mathrm{Aut}(\overline{S})$, since it is a smaller genus problem, and then construct A from $M \hookrightarrow A \twoheadrightarrow \overline{A}$. See Proposition 4.4.

(2) The genus of S is less than or equal to the critical genus. Then, we have to look for exceptional automorphisms (after finding the normalizer) in a finite number of cases, working as noted above.

Since strong branching simplifies the process of finding A, there is much potential for its use in determining full automorphism groups, possibly inductively as suggested by case 1. To date, strong branching has been used for a number of different families, with perhaps the most comprehensive use in determining full automorphism groups of cyclic p-gonal surfaces, see [22] and Subsection 5.1.1 (a surface is cyclic p-gonal when G has prime order).

Our main motivational goal in the following is to provide tools and techniques derived from the concept of strong branching to help classify full automorphism groups, and to provide explicit examples of how these techniques are used. We shall do this through first describing the general idea behind strong branching and surveying the current results in classification of automorphism groups that can be attributed to strong branching. Following this, we shall provide new classification results using strong branching, both single stage and inductively.

Outline of paper

The outline of our work is as follows. In Section 2 we covering preliminaries on branched coverings and ramification, the Riemann-Hurwitz theorem, group actions, and families of surfaces with a simultaneous group action. In Section 3 we provide details on how to determine whether or not an n-gonal group G extends to some larger automorphism group, providing very explicit results in certain special cases. In Section 4 we introduce strong branching, weakly normal actions and trivial core actions. In Section 5 we apply the concepts and methods of Sections 2 and 4, particularly strong branching, to finding full automorphism groups of families of n-gonal surfaces, surveying the known results and presenting new ones.

Acknowledgement

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2. Preliminaries

There are several tools for working with group actions on Riemann surfaces: Fuchsian groups, function fields, and branched covering theory. In this paper we use branched covering theory since strong branching and group actions are conveniently formulated in these terms. Moreover, these methods work in positive characteristic.

2.1. **Branched coverings and differentials.** Let S_1, S_2 be two Riemann surfaces of genus σ_1 and σ_2, respectively, and $\pi : S_1 \to S_2$ a branched covering (holomorphic map) of degree n. Some items related to the map π, useful in understanding the Riemann Hurwitz formula are:

(1) The differential map on tangent bundles

$$d\pi : T_P(S_1) \to T_{\pi(P)}(S_2)$$

and its dual pullback map of meromorphic differential 1-forms

$$d\pi^* : \Omega^1(S_2) \to \Omega^1(S_1).$$

(2) A divisor $(d\pi)$ defined on S_1 by

$$(d\pi) = \sum_{P \in S_1} ord_P(d\pi)P.$$

The value $ord_P(d\pi)$ is computed by first writing, in local coordinates centered at 0 in the domain and target,

$$\pi(z) = z^{e(P)} f(z),\ \ f(z) \neq 0.$$

Then

$$d\pi = z^{e(P)-1}(e(P)f(z)dz + zdf(z)).$$

Since

$$e(P)f(z)dz + zdf(z) = e(P)f(0)dz$$

at $z = 0$ then $ord_P(d\pi) = e(P) - 1$. Now $e(P) \geq 1$ for all P, it is independent of the coordinatization, and $e(P) > 1$ for at most finitely many points. Thus the divisor $(d\pi)$ of the differential $d\pi$ is given by

$$(d\pi) = \sum_{P \in S_1} (e(P) - 1)\, P. \tag{1}$$

2.2. Ramification and the Riemann-Hurwitz equation.

Definition 2.1. The *total ramification* of a branched covering π is the degree of the divisor in equation (1):

$$R_\pi = \sum_{P \in S_1} (e(P) - 1)\,. \tag{2}$$

If ω is a differential form on S_2 then the degree of the divisor $(d\pi^*(\omega))$ may be computed in two ways: first as a differential form on S_1 with degree $2(\sigma_1 - 1)$ and, secondly, as the degree of the pullback $d\pi^*(\omega)$ to get $2n(\sigma_2 - 1) + \sum_{P \in S_1} (e(P) - 1)$. The first term comes from pulling back the zeros and poles of ω and the second term comes from the ramification of the branched covering. The Riemann-Hurwitz equation may then be written:

$$2(\sigma_1 - 1) = 2n(\sigma_2 - 1) + \sum_{P \in S_1} (e(P) - 1) \tag{3}$$

or

$$2(\sigma_1 - 1) - 2n(\sigma_2 - 1) = R_\pi. \tag{4}$$

Note that we may use equation (4) to compute either σ_1, σ_2 or n. Specifically, for the index we must have:

$$n = \frac{2(\sigma_1 - 1) - R_\pi}{2(\sigma_2 - 1)}. \tag{5}$$

If $Q_1, \ldots, Q_t$ are the points on S_2 over which π is ramified, then another version of the Riemann-Hurwitz equation which emphasizes this branching is:

$$R_\pi = \sum_{P \in S_1} (e(P) - 1) = \sum_{j=1}^{t} \sum_{\pi(P) = Q_j} (e(P) - 1)\,.$$

Now $\sum_{\pi(P) = Q_j} (e(P) - 1) = n - \left|\pi^{-1}(Q_j)\right|$, so that we also have:

$$R_\pi = n \sum_{j=1}^{t} \left(1 - \frac{\left|\pi^{-1}(Q_j)\right|}{n}\right). \tag{6}$$

It follows that if we can count singular preimages, the total ramification is easily calculated.

2.3. Group actions, generating vectors, and signatures.

We now survey the main tools we need to describe group actions on surfaces.

Actions and surface kernel epimorphisms

The finite group G *acts conformally* on the Riemann surface S if there is a monomorphism:

$$\epsilon : G \hookrightarrow \mathrm{Aut}(S).$$

When there is no confusion we will identify G with it image $\epsilon(G)$. Such actions of G allow us to construct surfaces and analyze their automorphism groups with the group G as the starting point. Our primary tool for working with actions are surface kernel epimorphisms and the corresponding generating vectors, which we proceed to define.

The quotient surface $S/G = T$ is a closed Riemann surface of genus τ with a unique conformal structure such that

$$\pi_G : S \to S/G = T \tag{7}$$

is holomorphic. The quotient map $\pi_G : S \to T$ is ramified uniformly (all branching orders are the same on a given fiber) over a finite set $B_G = \{Q_1, \ldots, Q_t\}$ such that π_G is an unramified covering exactly over $T^\circ = T - B_G$. Let $S^\circ = \pi_G^{-1}(T^\circ)$ so that $\pi_G : S^\circ \to T^\circ$ is an unramified covering space whose group of deck transformation equals $\epsilon(G)$, restricted to S°. This covering determines a normal subgroup $\Pi_G = \pi_1(S^\circ) \lhd \pi_1(T^\circ)$ and an exact sequence $\Pi_G \hookrightarrow \pi_1(T^\circ) \twoheadrightarrow \epsilon(G)$ by mapping loops to deck transformations, via path lifting. Combine the last map with $\epsilon(G) \overset{\epsilon^{-1}}{\to} G$ to get an exact sequence

$$\Pi_G \hookrightarrow \pi_1(T^\circ) \overset{\xi}{\twoheadrightarrow} G. \tag{8}$$

The map ξ, which we call a *surface kernel epimorphism,* is an analogue to surface kernel epimorphisms for Fuchsian groups. The map ξ is well-defined only up to automorphisms of G. We detail this dependence and some questions related to computations with ξ at the end of this subsection.

Generating systems and generating vectors

The fundamental group $\pi_1(T^\circ)$ has presentation:

$$\left\{\alpha_i, \beta_i, \gamma_j, 1 \le i \le \tau, 1 \le j \le t \,\middle|\, \prod_{i=1}^{\tau}[\alpha_i, \beta_i] \prod_{j=1}^{t} \gamma_j = 1\right\}. \tag{9}$$

We denote the ordered generating set $(\alpha_1, \ldots, \alpha_\tau, \beta_1, \ldots, \beta_\tau, \gamma_1, \ldots, \gamma_t)$ by $\mathcal{G}$, noting that it is not unique.

Define

$$a_i = \xi(\alpha_i), b_i = \xi(\beta_i), c_j = \xi(\gamma_j).$$

The $2\tau + t$ tuple

$$\mathcal{V} = (a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t) \tag{10}$$

is called a *generating vector* for the action. We observe that

$$G = \langle a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t \rangle, \tag{11}$$

as ξ is surjective. Since the element c_j generates the stabilizer of some point P_j lying over Q_j, we have:

$$o(c_j) = n_j, \tag{12}$$

the ramification degree at P_j. Finally, the relation in (9), combined with equation (12), shows that a generating vector satisfies the following relations:

$$\prod_{i=1}^{\tau}[a_i, b_i] \prod_{j=1}^{t} c_j = c_1^{n_1} = \cdots = c_t^{n_t} = 1. \tag{13}$$

The *signature* of the action – actually of the generating vector – is $(\tau; n_1, \ldots, n_t)$. For conciseness, we call the "vector" $\mathcal{V}$ given in equation (10) a $(\tau; n_1, \ldots, n_t)$-generating vector of G. We call the number τ (the genus of S/G) the *orbit genus* and the numbers $n_1, \ldots, n_t$ the *periods* of the signature. In the n-gonal case with $\tau = 0$ we write $(n_1, \ldots, n_t)$. By the orbit-stabilizer theorem, $|G| = n_j \left|\pi_G^{-1}(Q_j)\right|$. Therefore, when the action of a group G on a compact Riemann surface S of genus σ is described using the signature $(\tau; n_1, \ldots, n_t)$ the Riemann-Hurwitz formula can be rewritten as a genus formula:

$$\sigma = 1 + n(\tau - 1) + \frac{n}{2} \sum_{j=1}^{t} \left(1 - \frac{1}{n_j}\right), \tag{14}$$

or the area of a fundamental domain

$$\frac{\text{Area}(S/G)}{2\pi} = \frac{2\sigma - 2}{|G|} = (2\tau - 2) + \sum_{j=1}^{t} \left(1 - \frac{1}{n_j}\right). \tag{15}$$

Any $2\tau + t$ tuple of elements of G satisfying conditions (11)-(13) is called a $(\tau; n_1, \ldots, n_t)$-generating vector, even though it may not have arisen from a G action. However, all such arbitrary generating vectors do arise from surfaces with a G action. We state this as a proposition and show the construction in the proof sketch.

Proposition 2.1. *Suppose we are given a surface T of genus τ, a branch set $B_G = \{Q_1, \ldots, Q_t\} \subset T$, $Q_0 \in T^\circ = T - B_G$ and generating set $\mathcal{G}$ of $\pi_1(T^\circ, Q_0)$ as given in (9). Then, given an arbitrary generating vector $\mathcal{V}$, as in equation (10), with signature $(\tau; n_1, \ldots, n_t)$ we may construct a surface S with G action such that $S/G = T$, π_G is branched over B_G, and such that $\mathcal{V}$ is the generating vector of the action.*

Proof. Using the generating vector $\mathcal{V}$ we can construct a surface kernel epimorphism $\Pi_G \hookrightarrow \pi_1(T^\circ, Q_0) \overset{\xi}{\twoheadrightarrow} G$. The subgroup Π_G defines a holomorphic unbranched covering of $S^\circ \to T^\circ$ with deck group G. Using the Riemann removable singularity theorem we can close up S° and T° to a branched covering $S \to T$ with G action. □

Example 2.1. If G is cyclic of order 7 with generator x, then (x, x, x^5) is a $(7, 7, 7)$-generating vector for G. Using the Riemann-Hurwitz formula, we see that we get a G action with signature $(7, 7, 7)$ on a surface of genus 3.

In the case of n-gonal actions, the primary focus of this paper, we only have the generators $\gamma_1, \ldots, \gamma_t$. We need to describe $\gamma_1, \ldots, \gamma_t$ so that we may compute the action of conformal maps upon them.

Construction 2.2. *Such a system may be constructed as follows.*

(1) *Select a system of arcs from the base point Q_0 to the Q_j so that the arcs only intersect at Q_0.*
(2) *Moreover, in a small neighborhood of Q_0 the counterclockwise order of the arcs is determined by the given order $Q_1, \ldots, Q_t$ of the end points.*
(3) *To construct γ_j we start out from Q_0 along the arc to Q_j, stopping just short of Q_j, encircling Q_j counterclockwise once in a small circle centered at Q_j, and then return to Q_0 along the initial path.*

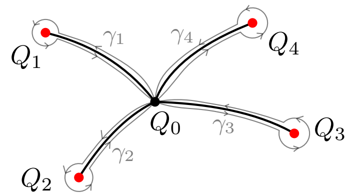

FIGURE 1. Construction 2.2.

It follows from the construction that the γ_j generate the group and that $\gamma_1 \cdots \gamma_t = 1$. See Figure 1.

Dependence on base points

We have left out base points to simplify the exposition, and so ξ is ambiguous up to inner automorphisms. First suppose that $Q_0 \in T^\circ$, and that path lifting $\pi_1(T^\circ) \twoheadrightarrow \epsilon(G)$ is defined with respect to the point P_0 lying over Q_0. If another point gP_0 is selected and ξ' is the new surface kernel epimorphism then

$$\xi' = Ad_g \circ \xi, \tag{16}$$

where $Ad_g(x) = gxg^{-1}$. Next, given two base points $Q_0, Q_0' \in T^\circ$ and a path δ from Q_0 to Q_0', the loop concatenation map $\varphi_\delta : \pi_1(T^\circ, Q_0) \to \pi_1(T^\circ, Q_0')$, $\varphi_\delta : \alpha \to \delta^{-1} * \alpha * \delta$ is an isomorphism unique up to an inner automorphisms of $\pi_1(T^\circ, Q_0)$ and $\pi_1(T^\circ, Q_0')$. For, if δ_1, δ_2 are two different paths Q_0 to Q_0' then

$$\varphi_{\delta_2} = \varphi_{\delta_1} \circ Ad_{\delta_2 * \delta_1^{-1}}$$

and

$$\varphi_{\delta_2} = Ad_{\delta_1^{-1} * \delta_2} \circ \varphi_{\delta_1}.$$

Now suppose that ξ and ξ' are defined with respect to $P_0 \in \pi_G^{-1}(Q_0)$ and $P_0' \in \pi_G^{-1}(Q_0')$, $\widetilde{\delta}$ is a path from P_0 to P_0' in S°, and $\delta = \pi_G(\widetilde{\delta}\ )$. Then

$$\xi = \xi' \circ \varphi_\delta. \tag{17}$$

If δ' is any other path from Q_0 to Q_0' then

$$\begin{aligned} \xi' \circ \varphi_{\delta'} &= \xi' \circ Ad_{\delta^{-1} * \delta'} \circ \varphi_\delta \\ &= Ad_{\xi'(\delta^{-1} * \delta')} \circ \xi' \circ \varphi_\delta \\ &= Ad_{\xi'(\delta^{-1} * \delta')} \circ \xi. \end{aligned} \tag{18}$$

Action on generating vectors

Generating vectors for actions are not unique. We may first apply any automorphism ω of G to the G action ϵ to obtain $\omega \circ \epsilon$. The result $\mathcal{V} \to \omega\mathcal{V}$ on the generating vector is

$$(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t) \to (\omega a_1, \ldots, \omega a_\tau, \omega b_1, \ldots, \omega b_\tau, \omega c_1, \ldots, \omega c_t).$$

The action of an automorphism does not affect the surface constructed from the generating vector since the subgroup Π_G is not affected by ω. This is consistent with our observations on the dependence on base points in equations (16),(17), and (18).

Secondly, we may use a different generating set $\mathcal{G}'$, and in turn this change of generating set has an effect on generating vectors. In the n-gonal case it can be shown that any such transformation $\mathcal{G} \to \mathcal{G}'$ has the form

$$\gamma_j \to \psi_j \gamma_{\theta(j)} \psi_j^{-1} \tag{19}$$

where ψ_j is a word in $\gamma_1, \ldots, \gamma_t$, and θ is a permutation of $1, \ldots, t$. The action on the generating vectors is

$$c_j \to w_j c_{\theta(j)} w_j^{-1}, \tag{20}$$

where w_j is obtained by replacing γ_i by c_i, for all i, in ψ_j. In the Abelian case the transformation is given by

$$c_j \to c_{\theta(j)}. \tag{21}$$

We call the actions given by equations (19) and (20) *braid actions.* Any two generating vectors $(c_1, \ldots, c_t)$ and $(c_1', \ldots, c_t')$ are called *braid equivalent* if $c_j' = w_j c_{\theta(j)} w_j^{-1}$ under the braid action. If θ is trivial we say that the vectors are *pure braid equivalent.* The origin of the term braid action is given in Remark 2.1. The braid action on the surfaces lying over (T, B_G) is discussed at the end of Section 2.5.

Also see [16] for more on the braid action.

Remark 2.1. Here is the connection to braid groups and the justification for calling the action in (19) and (20) the braid action. We may continuously move one branch set $\{Q_1, \ldots, Q_t\}$ to another via a path $(Q_1(s), \ldots, Q_t(s)), 0 \le s \le 1$, with $(Q_1(0), \ldots, Q_t(0)) = (Q_1, \ldots, Q_t)$ By standard theory, there is a family of homeomorphisms

$$h_s : T - \{Q_1, \ldots, Q_t\} \to T - \{Q_1(s), \ldots, Q_t(s)\}.$$

If $\{Q_1(1), \ldots, Q_t(1)\} = \{Q_1, \ldots, Q_t\}$ as sets then the homeomorphism h_1 is a homeomorphism of T° inducing the transformations in equations (19) and (20). The path

$$(Q_1(s), \ldots, Q_t(s)), 0 \le s \le 1$$

with $\{Q_1(1), \ldots, Q_t(1)\} = \{Q_1, \ldots, Q_t\}$ is a braid and hence we use the term braid action.

2.4. **Generating vectors and signatures of subgroups.** Our main approach to determining the full automorphism group of a surface will be to start with a group which we know acts on a surface, and then see if it extends to a larger group. Accordingly, we need to know how signatures of groups are related to their subgroups. Fortunately, once a G action has been specified via a $(\tau; n_1, \ldots, n_t)$-generating vector, we can recover the signature of a subgroup $G \le A$ using the following theorem of Singerman [20].

Theorem 2.3. *For a group A, given a $(\tau_A; n_1, \ldots, n_t)$-generating vector*

$$(a_1, \ldots, a_{\tau_A}, b_1, \ldots, b_{\tau_A}, c_1, \ldots, c_t)$$

for A, the signature of the subgroup G with index d is

$$(\tau_G; m_{1,1}, m_{1,2}, \ldots, m_{1,\theta_1}, \ldots m_{t,\theta_t})$$

where

(1) *If $\Phi\colon A \to S_d$ is the permutation representation of A on the cosets of G, then the permutation $\Phi(c_j)$ has precisely θ_j cycles of length less than n_j, the lengths of these cycles being*

$$n_j/m_{j,1}, \ldots n_j/m_{j,\theta_j}.$$

(2) *The index d satisfies*

$$d = \frac{2\tau_G - 2 + \sum_{j=1}^{t}\sum_{i=1}^{\theta_j}\left(1 - \frac{1}{m_{j,i}}\right)}{2\tau_A - 2 + \sum_{j=1}^{t}\left(1 - \frac{1}{n_j}\right)}. \tag{22}$$

When G is normal in A, so that $A = N$, the cycles of $\Phi(c_j)$ all have the same length. Thus by considering the action of N/G on S/G, Theorem 2.3 can be simplified to:

Proposition 2.4. *For a group N, given a $(\tau_N; n_1, \ldots, n_t)$-generating vector*

$$(a_1, \ldots, a_{\tau_N}, b_1, \ldots, b_{\tau_N}, c_1, \ldots, c_t)$$

for N, the signature of the normal subgroup G of index d is

$$(\tau_G; m_{1,1}, m_{1,2}, \ldots, m_{1,\theta_1}, \ldots m_{t,\theta_t})$$

where:

(1) *$m_{j,i} = n_j/l_j$ and $\theta_j = d/l_j$ where l_j is the order of c_jG in N/G, and*
(2) *the index d satisfies*

$$d = \frac{2\tau_G - 2}{2\tau_N - 2 + \sum_{j=1}^{t}\left(1 - \frac{1}{l_j}\right)}. \tag{23}$$

Remark 2.2. Let A be a group acting on the surface S with signature $(\tau_A; n_1, \ldots, n_t)$. Let $G < A$ act on S with signature $(\tau_G; m_1, \ldots, m_s)$. We can compute the index $d = |A| / |G|$ without knowing the structure of $S/G \to S/A$. Specifically, as in Theorem 2.3, or using equation (15),we have

$$d = |A| / |G| = \frac{(2\sigma - 2)/|G|}{(2\sigma - 2)/|A|} = \frac{2\tau_G - 2 + \sum_{j=1}^{s}\left(1 - \frac{1}{m_j}\right)}{2\tau_A - 2 + \sum_{j=1}^{t}\left(1 - \frac{1}{n_j}\right)}. \tag{24}$$

Remark 2.3. Using Theorem 2.3, a MAGMA script can be written that takes a finite group A and a generating vector $\mathcal{V} = (c_1, \ldots, c_t)$ and computes the genus σ of the surface S, defined by A and $\mathcal{V}$ and the signature of the action for every subgroup $G \leq A$. We use this script to look for interesting n-gonal subgroup actions given a proposed full automorphism group.

2.5. **Equivalence, families, and equisymmetry of actions.** When trying to extend the known action of a n-gonal group G to a larger, normalizing group, the notion of conformal equivalence of actions naturally arises, specifically the diagram (26). In turn, this leads to looking for relations among the branch points on the quotient surface. The notion of strong branching, to be discussed in the next section, automatically forces an action to have numerous branch points. By varying the branch points we get families of surface with the "same" action. Thus, in our quest to classify surface automorphism groups, it is useful to introduce the interrelated, clarifying concepts of conformal equivalence of actions, families of actions, and equisymmetry of actions.

Equivalence of actions

Two actions ϵ_1, ϵ_2 of G on possibly different surfaces S_1, S_2 are conformally equivalent if there is an equivariant, conformal homeomorphism $h : S_1 \to S_2$ and an automorphism $\omega \in \mathrm{Aut}(G)$ such that $h\epsilon_1(\omega(g)) = \epsilon_2(g)h$, or more conveniently:

$$\epsilon_2(g) = h\epsilon_1(\omega(g))h^{-1}, \forall g \in G. \tag{25}$$

The conformal map $h : S_1 \to S_2$ induces a conformal map $\overline{h} : T_1 \to T_2$, and in diagram form we have:

$$\begin{tikzcd} S_1 \arrow[r, "h"] \arrow[d, "\pi_{G_1}"] & S_2 \arrow[d, "\pi_{G_2}"] \\ T_1 \arrow[r, "\overline{h}"] & T_2 \end{tikzcd} \tag{26}$$

where G_1 and G_2 denote the subgroups $\epsilon_1(G) \le \mathrm{Aut}(S_1)$, $\epsilon_2(G) \le \mathrm{Aut}(S_2)$. The conformal homeomorphism $\overline{h} : T_1 \to T_2$ must preserve branch points and their orders and hence defines a conformal homeomorphism $T_1^\circ \to T_2^\circ$. Frequently, we shall start with the bottom of diagram (26) given and want to fill in the top.

We start our discussion with the following proposition expressing conformal equivalence in terms of generating vectors.

Proposition 2.5. *Let S_1 and S_2 be surfaces, and let $G_1 \le \mathrm{Aut}(S_1)$ and $G_2 \le \mathrm{Aut}(S_2)$ be subgroups (that are not initially assumed to be of the form $\epsilon_1(G)$ and $\epsilon_2(G)$), but satisfy diagram (26). Let T_1 and T_2, be the respective quotients. Also, let*

$$\mathcal{G} = \{\alpha_1, \ldots, \alpha_\tau, \beta_1, \ldots, \beta_\tau, \gamma_1, \ldots, \gamma_t\}$$

be a generating system for $\pi_1(T_1^\circ, Q_0)$ and $(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t)$ the corresponding generating vector for G_1, determined by a point P_0, lying over Q_0. Then the following hold:

(1) *The group $G_2 = hG_1h^{-1}$ and hence $G_1 = \epsilon_1(G)$ and $G_2 = \epsilon_2(G)$ for a common group G acting on S_1 and S_2.*
(2) *The map $\overline{h}$ maps the branch points of π_{G_1} to branch points of π_{G_2} of the same order. Hence $\overline{h} : T_1^\circ \to T_2^\circ$ is a conformal homeomorphism.*
(3) *Let*

$$\mathcal{G}' = \{\alpha_1', \ldots, \alpha_\tau', \beta_1', \ldots, \beta_\tau', \gamma_1', \ldots, \gamma_t'\}$$

be the generating system for $\pi_1(T_2^\circ, \overline{h}(Q_0))$ obtained by applying $\overline{h}$ to $\mathcal{G}$, and $(a_1', \ldots, a_\tau', b_1', \ldots, b_\tau', c_1', \ldots, c_t')$ the generating vector of G_2 derived from $\mathcal{G}'$ at

the point $h(P_0)$. Then

$$a_i' = ha_ih^{-1}, b_i' = hb_ih^{-1}, c_j' = hc_jh^{-1} \tag{27}$$

for all i and j.

Proof. To see statement 1, observe that h maps G_1 orbits to G_2 orbits, namely $h(G_1P) = G_2h(P)$ for all $P \in S_1$. For any $P \in S_1$, and $g \in G_1$

$$h(g(P)) = g'(h(P))$$

for some $g' \in G_2$. Setting $P = h^{-1}(P')$ we get

$$h(g(h^{-1}(P'))) = g'(h(h^{-1}(P'))) = g'(P').$$

It follows that $hgh^{-1} \in G_2$. Thus $g \to hgh^{-1}$ maps G_1 to G_2 with inverse $g' \to h^{-1}g'h$.

By statement 1, we observe that $|G_1| = |G_2|$. For statement 2, observe that the branching order at a point $Q = \pi_{G_1}(P)$ equals $|G_1| \,/\, \left|\pi_{G_1}^{-1}(Q)\right| = |G_2| \,/\, \left|\pi_{G_2}^{-1}(\overline{h}(Q))\right|$, so branching orders are preserved.

For equation 3, let P_0 lie over Q_0, $\alpha \in \pi_1(T_1^\circ, Q_0)$ and let $\widetilde{\alpha}$ be the lift of α to S_1° that is based at P_0. The lift of $\overline{h}(\alpha)$ to S_2° starting at $h(P_0)$ will be $h(\widetilde{\alpha})$. Thus $\xi'(\overline{h}(\alpha))$ is the element $x \in G_2$ such that

$$\begin{aligned} h(\widetilde{\alpha})(1) &= x(h(P_0)), \\ h(\xi(\alpha)P_0) &= x(h(P_0)), \\ h\xi(\alpha)h^{-1} &= x. \end{aligned}$$

This establishes criterion (27). □

Now, assume that the bottom and sides of the diagram (26) are given. If we want to fill in the top as in diagram (28), where the map, h, to be filled in is denoted by a dashed arrow, we need a criterion that, when satisfied, guarantees the existence of the covering transformation h.

(28)
$$\begin{tikzcd} S_1 \arrow[r, dashed, "h"] \arrow[d, "\pi_{\epsilon_1(G)}"] & S_2 \arrow[d, "\pi_{\epsilon_2(G)}"] \\ T_1 \arrow[r, "\overline{h}"] & T_2 \end{tikzcd}$$

Proposition 2.6. *Suppose that we have two actions ϵ_1, ϵ_2 of the same group G on two surfaces S_1, S_2 as diagram (28). Suppose further that $\overline{h}$ is a conformal homeomorphism, and that h is a map to be found as indicated by the dotted line. We also assume that:*

(1) *The map $\overline{h}$ maps the branch points of $\pi_{\epsilon_1(G)}$ to branch points of $\pi_{\epsilon_2(G)}$ of the same order. Hence $\overline{h} : T_1^\circ \to T_2^\circ$ is a conformal homeomorphism.*
(2) *Let*
$$\mathcal{G} = \{\alpha_1, \ldots, \alpha_\tau, \beta_1, \ldots, \beta_\tau, \gamma_1, \ldots, \gamma_t\}$$
be a generating system for $\pi_1(T_1^\circ, Q_0)$ and $(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t)$ the corresponding generating vector of G obtained from $\mathcal{G}$ and a specific P_0 lying over Q_0. Let $Q_0' = \overline{h}(Q_0)$, $P_0' \in \pi_{\epsilon_2(G)}^{-1}(Q_0')$ and
$$\mathcal{G}' = \{\alpha_1', \ldots, \alpha_\tau', \beta_1', \ldots, \beta_\tau', \gamma_1', \ldots, \gamma_t'\}$$
be the generating system for $\pi_1(T_2^\circ, Q_0')$ obtained by applying $\overline{h}$ to $\mathcal{G}$, and $(a_1', \ldots, a_\tau', b_1', \ldots, b_\tau', c_1', \ldots, c_t')$ the generating vector of G derived from $\mathcal{G}'$, with lifting starting at P_0'.

Then there exists an invertible conformal map h as in diagram (28) with $h(P_0) = P_0'$, if and only if there is a automorphism ω of G such that

(29)
$$a_i' = \omega(a_i), b_i' = \omega(b_i), c_j' = \omega(c_j).$$

for all i and j.

Proof. If $h : S_1 \to S_2$ exists, completing the diagram (28), then as we saw in Proposition 2.5 the automorphism ω is induced by conjugation by h, pulled back to G.

For the other direction let us assume that the criterion (29) holds and prove that h exists. From covering space theory, the map h exists (with the branch points and preimages removed) if and only if

$$\overline{h}_* \circ \left(\pi_{\epsilon_1(G)}\right)_* (\pi_1(S_1^\circ, P_0)) = \left(\pi_{\epsilon_2(G)}\right)_* (\pi_1(S_2^\circ, P_0')).$$

Consider the diagram

$$\begin{tikzcd}
\pi_1(S_1^\circ, P_0) \arrow[r, dashed, "\overline{h}_*"] \arrow[d, "\left(\pi_{\epsilon_1(G)}\right)_*"] & \pi_1(S_2^\circ, P_0') \arrow[d, "\left(\pi_{\epsilon_2(G)}\right)_*"] \\
\pi_1(T_1^\circ, Q_0) \arrow[r, "\overline{h}_*"] \arrow[d, "\xi"] & \pi_1(T_2^\circ, Q_0') \arrow[d, "\xi'"] \\
G \arrow[r, "\omega"] & G
\end{tikzcd}$$

We are proposing that putting the map $\overline{h}_*$ (suitably restricted) into the top row gives a commutative diagram. In particular, we need to prove that the image is as suggested. The only arrow in question is the top row, indicated by the dashed arrow. The subdiagram formed from the bottom two rows is commutative since the commutativity requirement holds for every element of the generating set $\mathcal{G}$ of $\pi_1(T_1^\circ, Q_0)$, according to equation (29). Furthermore, the horizontal maps are isomorphisms and the vertical maps are surjections. Now consider the subdiagram formed from the top two rows. The vertical maps are injective because the columns of diagram (28) are covering spaces. Since the columns of the entire diagram are exact and the bottom subdiagram commutes then $\overline{h}_*$ in the top row maps the kernels isomorphically as suggested. Thus, by covering space theory, we have constructed a partial map $h : S_1^\circ \to S_2^\circ$. As shown in the proof of Proposition 2.1, the map h may be completed to a conformal homeomorphism $h : S_1 \to S_2$ satisfying the requirements. □

Remark 2.4. If we allow h in equation (25) to be just a homeomorphism then the actions are said to be *topologically equivalent*. For a given genus there are only finitely many topological equivalence classes. For more detail see [6] and [7].

Remark 2.5. Suppose we have fixed a quotient surface T, (ordered) branch set $B_G = \{Q_1, \ldots, Q_t\}$, and signature $\mathcal{S} = (\tau, n_1, \ldots, n_t)$. Once we have fixed a generating set $\mathcal{G} \subset \pi_1(T^\circ, Q_0)$ we can enumerate the surfaces $S \to T$ and actions $\epsilon : G \to \mathrm{Aut}(S)$ with the given T, B_G, $\mathcal{S}$ by means of generating vectors. The actions are in $1-1$ correspondence with the generating vectors. The automorphism group $\mathrm{Aut}(G)$ acts freely on the generating vectors. Each $\mathrm{Aut}(G)$ class of vectors determines a unique branched covering space $S \to T$ with G-action and a unique subgroup $\epsilon(G) \leq \mathrm{Aut}(S)$. Two such coverings $S_1 \to T$, $S_2 \to T$, are equivalent if and only if the diagram (28) can be completed with $T_1 = T_2$ and $\overline{h} \in \mathrm{Aut}(T, B_G, \mathcal{S})$, the group of conformal automorphism of T respecting the branch points and signature. Thus the set of all covers $S \to T$ and equivalence classes of actions $\epsilon : G \to \mathrm{Aut}(S)$ are the equivalence classes of generating vectors under the action of $\mathrm{Aut}(T, B_G, \mathcal{S}) \times \mathrm{Aut}(G)$. For more detail see [6] and [7].

Families of curves and equisymmetry

Special placement of the branch points allows for extra automorphisms beyond the action of G. For instance Shaska [19] determines which hyperelliptic curves have extra automorphisms by means of equations in the coefficients of the defining equations of hyperelliptic curves. In [16] Magaard, Shaska, Shpectorov, and Völklein discuss families of curves in moduli space and the links to the braid action and Hurwitz spaces. Our notion of family is very informal and is closer to a Hurwitz space than the equisymmetric strata of the branch locus of moduli space, discussed in [6]. Our definition will allow for curves in positive characteristic, so we use the term curve instead of surface. The example of cyclic n-gonal curves, in Section 5.1, is a simple tractable example.

A *family of curves* $\{S_b : b \in B\}$ is a morphism $\pi : E \to B$ such that each $S_b = \pi^{-1}(b)$, $b \in B$ is a smooth closed curve (compact Riemann surface). We assume that B is an irreducible variety or connected manifold. A *family of actions* for a family of smooth curves $\pi : E \to B$ is a family of monomorphisms

$$\epsilon_b : G \to \mathrm{Aut}(\pi^{-1}(b)),\ b \in B$$

such that: for each $g \in G$ the map $(b, x) \to (b, \epsilon_b(g)x)$ is an automorphism of the variety (manifold) $V = \{(b, x) : \pi(x) = b\}$. In [12], Guerrero discusses an expanded version of families of curves by using *holomorphic families of curves* where now the map $\pi : E \to B$ is holomorphic and B is a connected, complex manifold.

We also allow holomorphic families, since it is useful in studying the moduli space and Teichmüller space of surfaces. However, in the positive characteristic case, B must be an irreducible, locally-closed variety.

Two actions $\epsilon_1 : G \to \mathrm{Aut}(S_1)$ and $\epsilon_2 : G \to \mathrm{Aut}(S_2)$ of G on S_1 and S_2 are *(directly) equisymmetric* $\epsilon_1 \sim_D \epsilon_2$ if there is a family of curves $\pi : E \to B$ with a family of actions $\epsilon_b : G \to \mathrm{Aut}(\pi^{-1}(b))$, $b \in B$ such that there are $b_1, b_2 \in B$ with isomorphisms $\phi_i : \pi^{-1}(b_i) \backsimeq S_i$ and $\epsilon_i = \phi_i \circ \epsilon_{b_i} \circ \phi_i^{-1}$. Two actions $\epsilon_1 : G \to \mathrm{Aut}(S_1)$ and $\epsilon_m : G \to \mathrm{Aut}(S_m)$ are equisymmetric if there is a sequence of surfaces S_i and actions $\epsilon_i : G \to \mathrm{Aut}(S_i)$ such that $\epsilon_1 \sim_D \epsilon_2$, $\epsilon_2 \sim_D \epsilon_3, \ldots, \epsilon_{m-1} \sim_D \epsilon_m$. Typically the relations $\epsilon_i \sim_D \epsilon_{i+1}$ come from distinct families as i varies.

Remark 2.6. It is possible that two G actions are equisymmetric without the automorphism groups of the surfaces being isomorphic. In such a case we may have $\epsilon_i(G) \lneq \mathrm{Aut}(S_i)$ even though $\epsilon_b(G) = \mathrm{Aut}(\pi^{-1}(b))$ generically. In fact these are the very cases we are interested in.

More on the braid action

Now we want to consider the effect of a change in basis. The change of generating set $\mathcal{G} \to \mathcal{G}'$ for n-gonal actions over a fixed pair (T, B_G) induces an automorphism $\Phi : \pi_1(T^\circ, Q_0) \to \pi_1(T^\circ, Q_0)$. This induces a right action of $\mathrm{Aut}(\pi_1(T^\circ, Q_0))$ on generating vectors via the action on surface kernel epimorphisms given by

$$\xi \to \xi^\Phi = \xi \circ \Phi. \tag{30}$$

Now suppose that $\mathcal{G}$, $\mathcal{G}', \mathcal{V} = (c_1, \ldots, c_t)$, and $\mathcal{V}' = (c_1', \ldots, c_t')$ are related by

$$\mathcal{G}' = \Phi(\mathcal{G})$$

and

$$\xi = \xi(\mathcal{G}') = \xi(\Phi(\mathcal{G})) = \xi^\Phi(\mathcal{G}) = \mathcal{V}^\Phi. \tag{31}$$

The explicit equations, derived from (19) and (20), are

$$\gamma_j' = \Phi(\gamma_j) = \psi_j \gamma_{\theta(j)} \psi_j^{-1} \tag{32}$$

and

$$c'_j = c_j^\Phi = w_j c_{\theta(j)} w_j^{-1}. \tag{33}$$

The equations $\mathcal{G}' = \Phi(\mathcal{G})$ and $\mathcal{V}' = \mathcal{V}^\Phi$ simply say that the surface constructed from T, B_G, $\mathcal{G}'$, ξ, and $\mathcal{V}'$ is the same as the surface constructed from T, B_G, $\mathcal{G}$, ξ^Φ, and $\mathcal{V}'$. Thus we can restrict our attention to a single generating set $\mathcal{G}$. Two surfaces constructed in such a way will be called *braid companions*.

Proposition 2.7. *Let G, T, $B_G = \{Q_1, \ldots, Q_t\}$, $\mathcal{S}$, $\mathcal{G}$, and $\pi_1(T^\circ, Q_0) \overset{\xi}{\twoheadrightarrow} G$ be as defined above and held fixed. Then we have:*

1. *The surfaces S with G-action such that $S/G = T$ and $S \to T$ is branched over B with signature $\mathcal{S}$ are in 1-1 correspondence with the $\mathcal{S}$-generating vectors of G.*
2. *Let $\Phi \in \mathrm{Aut}(\pi_1(T^\circ, Q_0))$. Then Φ is induced by a homeomorphism h of T° and the generating vector of the G-action on the surface induced by $\xi \circ \Phi$ is $\mathcal{V}^\Phi$ defined by equation (33).*
3. *If the homeomorphism h above is orientation preserving then Φ is induced by a braid $(Q_1(s), \ldots, Q_t(s)), 0 \le s \le 1$ in $\mathbb{P}^1(\mathbb{C})^t$ as in Remark 2.1. Braid equivalent actions are equisymmetric.*
4. *The set of generating vectors $\{\mathcal{V}\}$ and the corresponding induced surfaces $\{S_\mathcal{V}\}$ with the given signature $\mathcal{S}$ consists of several orbits of the group $\mathrm{Aut}_\mathcal{S}(\pi_1(T^\circ, Q_0))$ where the subscript denotes the subgroup of automorphisms preserving the signature. Specifically the permutation θ in equation (33) should preserve the signature $\mathcal{S}$.*
5. *The braid action is generated by the following transformations.*

$$c'_{j+1} = c_j,\ c'_j = c_j c_{j+1} c_j^{-1},$$
$$c'_k = c_k,\ \text{otherwise}.$$

Proof. Statements 1, 2, and 4 follow from previous discussion. Statements 3 and 5 are well known from the literature [3]. □

3. FINDING AUTOMORPHISM GROUPS AND THEIR SIGNATURES FOR n-GONAL SURFACES

In this section we describe processes for determining automorphism groups of n-gonal surfaces by examining whether or not an n-gonal action of G extends to a larger group A. For some results on cyclic groups see [10]. These processes naturally break up into two cases depending on whether G is normal in A as suggested in the introduction. We deal with each case separately in the next two subsections. We approach the problem with two different methods depending on the chosen equivalence type: topological or conformal. We first briefly describe the methods and then give some details and examples in the next two subsections. In each subsection we first describe signature theorems that apply to Method 1 and then give some details about Method 2. In Section 5, Method 1 is extensively used.

Method 1: topological equivalence

The first method is "moduli free", namely we try to extend the G action up to topological equivalence. We are not too concerned about the actual configuration of B_G, just the associated signature $\mathcal{S}$.

For this method, we first find possible N and then possible A algebraically. In each case the structures of the inclusions $G \lhd N$ and $N < A$ and the given signature $\mathcal{S}$ of the G action restrict the possibilities for N and A and their signatures. Next, generating vectors for N and then A are sought, which is a purely computational problem. The action of A on a surface restricts to one of G, and the signature of the action of G can be computed by Theorem 2.3. If G is n-gonal then we compare its signature with $\mathcal{S}$. With more work (beyond the scope of this paper) we may compute a generating vector for the action of G by using the monodromy representation of A on A/G and compare the vectors in order to understand if they are topologically equivalent. In this paper we mainly focus the question of which signatures extend.

Method 2: conformal equivalence

Before starting we recall the definition of the core of a subgroup of a group. If $G < H$ the core of G in H is given by

$$\mathrm{Core}_H(G) = \bigcap_{x \in H} xGx^{-1}. \tag{34}$$

We say that G has a trivial core in H or $G < H$ is a trivial core pair if

$$\mathrm{Core}_H(G) = \{1\}. \tag{35}$$

In our second method we retain the information on B_G so when we extend, the extensions that are permissible depend on B_G. We keep on extending the action of G to larger groups in a stepwise fashion. Given the available computational tools, especially the primitive groups database, we use an inductive method with three cases. For $G < A$ consider any chain of subgroups

$$G = G_0 < G_1 < \cdots < G_s = A \tag{36}$$

where for each successive pair $G_j < G_{j+1}$ we have one of the following cases:

(1) Case 1: The subgroup $G_j \lhd G_{j+1}$.
(2) Case 2: The coset space G_{j+1}/G_j is a faithful, primitive action space for G_{j+1}, namely $\mathrm{Core}_{G_{j+1}}(G_j) = \{1\}$ and there are no intermediate groups $G_j < H < G_{j+1}$.
(3) Case 3: There is $\{1\} \lhd M < G_j$ with $M \lhd G_{j+1}$.

Any chain of groups can be refined into such a chain. Case 3 is the general case and Cases 1 and 2 are the missing extreme cases where the core is trivial or all of G_j. In Case 2 we want a primitive action space so that we can use the primitive groups database. The transitive group database could be used but it is too unwieldily and does not have the range of the primitive groups database.

Starting with G, a branch set B_G, signature $(0; n_1 \ldots, n_t)$, and generating vector $(c_1 \ldots, c_t)$ we construct successive extensions $G_j < G_{j+1}$. Assuming we have constructed an action of G_{j+1}, the map $\pi_{j+1} : S/G_j \to S/G_{j+1}$ is a rational map of $\mathbb{P}^1$ to itself and the branch set B_{G_j} lies over $B_{G_{j+1}}$ via π_{j+1}. Furthermore, to construct the action of G_{j+1}, a generating vector $\mathcal{V}_{j+1}$ for G_{j+1}, with signature $\mathcal{S}_{j+1}$, needs to be computed with respect to a generating set $\mathcal{G}_{j+1} \subset \pi_1(\mathbb{P}^1 - B_{G_{j+1}})$ that is compatible with the map π_{j+1}. We discuss the construction of the generating vector and action in the next two subsections. When we can no longer extend the chain we have found the automorphism group of S. The Hurwitz bound $H \leq 84\,(\sigma - 1)$

forces

$$|H| / |G| \le 42 \cdot \left(\sum_{j=1}^{t} \left(1 - \frac{1}{n_i}\right) - 2 \right), \tag{37}$$

so that the process terminates.

Remark 3.1. We note that the sequence (36) depends on the configuration of the branch set B_G. Typically it is difficult to precisely determine the branch set B_{G_j} and generating vector $\mathcal{V}_j$. The scope of this paper allows us say the following: we can find all extensions $G < H$, and all generating vectors for n-gonal actions of H on a surface S such that the signature of the restricted G action on S has the initial signature $(n_1 \ldots, n_t)$. In principal, the branch set B_H can be lifted all the way up to S/G to produce branch set B'_G, and likewise a generating set $\mathcal{G}'_0 \subset \pi_1(\mathbb{P}^1 - B_G)$ and generating vector $\mathcal{V}'_0$. Even if $B'_G = B_G$, the generating vectors $\mathcal{V}'_0$ and $\mathcal{V}$ may not be easily comparable since the original generating sets $\mathcal{G}$ and $\mathcal{G}'_0$ may not be equal. So we can say that there is a G action on a surface S' that "looks like" the original G-action and extends to H. More precisely in the general family of surfaces with G-action with a fixed signature $\mathcal{S}$ there is a subfamily where the action extends to H. Typically "looks like" will mean that S' and S will be braid companions.

Remark 3.2. Though not directly relevant to our work here, the papers [2] and [13] discuss the possible monodromy groups of rational maps $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Our maps $S/G \to S/A$ are such maps, so the cited works allow us to say general things about the extensions $G < A$.

3.1. The normal extension case.

3.1.1. *Platonic Groups.* Given an n-gonal group G, since G is normal in N, the group $K = N/G$ acts on the surface $S/G = \mathbb{P}^1$, so that K is a finite subgroup of $\mathrm{PSL}(2, \mathbb{C})$. All such groups, and their signatures, are well known:

Theorem 3.1. *Any finite $K \le \mathrm{PSL}(2, \mathbb{C})$ is isomorphic to one of C_k, D_k, A_4, S_4 or S_5 (C_k is cyclic group of order k and D_k dihedral group of order $2k$). The signatures for each such group are given in Table 1.*

Group	Signature
C_k	(k, k)
D_k	$(2, 2, k)$
A_4	$(2, 3, 3)$
S_4	$(2, 3, 4)$
A_5	$(2, 3, 5)$

TABLE 1. Groups of Automorphisms and Signatures of $\mathbb{P}^1$

Notation 3.2. *An orbit $K \cdot P$ is called singular if $K_P \neq \{1\}$ and is called regular if $K_P = \{1\}$.*

Thus the possible N's satisfy the short exact sequence

$$G \hookrightarrow N \twoheadrightarrow K$$

which can be solved for a given G and K.

3.1.2. *Signatures for N.* The possible signatures for a normal extension N can be recovered from G and K using Proposition 2.4. Specifically, we have the following, which is a generalization of [22, Proposition 4.1]:

Proposition 3.3. *Suppose the signature of $K = N/G$ is (d_1, d_2, d_3) (with d_3 deleted if $K = C_k$) and let $\mathcal{O}(G)$ denote the set of orders of elements in G.*

(1) *The signature of N is of the form*

$$(a_1d_1, a_2d_2, a_3d_3, m_1, \dots, m_s)$$

where $a_i \in \mathcal{O}(G)$ and $m_i \in \mathcal{O}(G)\backslash\{1\}$.

(2) *The signature of G is*

$$(\underbrace{a_1, \dots a_1}_{|K|/d_1-times}, \underbrace{a_2, \dots a_2}_{|K|/d_2-times}, \underbrace{a_3, \dots a_3}_{|K|/d_3-times}, \underbrace{m_1, \dots m_1}_{|K|-times}, \dots \underbrace{m_r, \dots m_s}_{|K|-times})$$

where any 1's are removed.

Technically speaking, the way Proposition 3.3 has been stated, we are starting with the signature for N and finding the signature for G. However, given a specific signature for G, it is not hard to see how to reverse this process to determine the possible K's which could extend G and the corresponding signatures for N. We illustrate with a example.

Example 3.1. Example 2.1 shows the cyclic group G of order 7 acts on a genus 3 surface with signature $(7, 7, 7)$. We determine the signatures of possible normal extensions.

First, since none of the periods are divisible by 2, we can only have $K = C_k$ for some k. This means that N has signature of the form $(a_1k, a_2k, 7)$ where $a_1, a_2 \in \{1, 7\}$. Since G has just three periods, we must have $k \leq 3$. When $k = 3$, we must have $a_1 = a_2 = 1$ and N has signature $(3, 3, 7)$. When $k = 2$, must have exactly one of a_1 or a_2 equal to 7, and N has signature $(2, 7 \cdot 2, 7)$.

We note that just because a given N and corresponding signature for N exist does not mean that an n-gonal group G extends to N acting on an n-gonal surface. Thus next we consider conditions on generating vectors for an n-gonal group G which ensures the extension to some larger group N.

3.1.3. *Normally extending actions by cyclic groups.* Fix a branch set $B_G = \{Q_1, \dots, Q_t\}$, a generating system $\mathcal{G} = (\gamma_1, \dots, \gamma_t)$, and signature $\mathcal{S} = (n_1, \dots, n_t)$. As in Remark 2.5, all possible n-gonal G actions with given B_G and $\mathcal{S}$ are determined by a generating vector $(c_1, \dots, c_t)$ with respect to $\mathcal{G}$. When classifying and analyzing actions via generating vectors, all vectors need to be computed with respect to the given $\mathcal{G}$. When trying to extend a given action with respect to a subgroup of $\mathrm{Aut}(T, B_G, \mathcal{S})$ (see Remark 2.5) we need to choose a $\mathcal{G}$ adapted to transformations in $\mathrm{Aut}(T, B_G, \mathcal{S})$. We now consider the simple case that an automorphism h of S normalizes the action of G, so $K = C_k = \langle \overline{h} \rangle$. Since h normalizes G we have the following diagram

$$\begin{array}{ccc} S & \xrightarrow{h} & S \\ \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_G} \\ T & \xrightarrow{\overline{h}} & T \end{array} \tag{38}$$

where $\overline{h}$ is the induced map. We will construct a set of loops in T° adapted to the action of $\overline{h}$ and compute the action.

Construction 3.4. *For the purpose of discussion, we may assume that $\overline{h} : z \to uz$ is a rotation where u is a kth root of 1. It follows then that the set B_G consists of possible singular $\langle \overline{h} \rangle$ orbits $\{0\}$ and/or $\{\infty\}$ and p regular orbits $\left\{z_i, \ldots u^{k-1}z_i\right\}$ for various distinct z_i in $\mathbb{C}^*$.*

(1) *Select a ray ℓ from 0 to ∞ that contains no point of B_G. The k transforms $u^j\ell$ of ℓ cut up $\mathbb{C}$ into k wedges $W_1, \ldots, W_k$, where W_j is the wedge bounded by $u^{j-1}\ell$ and $u^j\ell$. Each of the orbits $\left\{z_i, \ldots u^{k-1}z_i\right\}$ meets each wedge in a unique interior point $u^j z_i$. We assume that $z_i \in W_1$ for all i.*
(2) *Order the z_i so that $|z_1| \leq \cdots \leq |z_p|$.*
(3) *Next we draw a simple, smooth arc $\zeta(t)$, $0 \leq t \leq 1$, lying in W_1, that starts at z_1, ends at z_p and passes through all the intermediate z_i in order. Modify the arc ζ slightly so that z_i lies slightly to the right of the curve as we traverse from start to finish.*
(4) *Select a point Q_0 on ℓ with $0 < |Q_0| < |z_1|$.*
(5) *We construct a series of p loops $\gamma_{i,1}$ defined as follows:*
    (a) *Follow a path from Q_0 to $\zeta(0)$ (the same path for each z_i).*
    (b) *Follow a path from $\zeta(0)$ to a point $\zeta(t_i)$ very near z_i. Pick $t_1 = 0$, $t_p = 1$, and the other t_i increasing in value.*
    (c) *Make a short excursion from $\zeta(t_i)$ towards z_i.*
    (d) *Make a small counterclockwise circle that lies entirely to the left of ζ.*
    (e) *After circling z_i return to Q_0 reversing the steps in a,b,c.*
(6) *The transformation $z \to u^{j-1}z$ maps W_1 to W_j and maps $\gamma_{i,1}$ to $u_*^{i-1}(\gamma_{i,1})$ Let δ_j be the counterclockwise arc from Q_0 to $u^{j-1}Q_0$ along the circle $|z| = |Q_0|$.*
(7) *Define*
$$\gamma_{i,j} = \delta_j u_*^{j-1}(\gamma_{i,1})\,\delta_j^{-1}.$$
(8) *Let γ_1 be the arc that travels along ℓ towards 0 encircles 0 in a small circle about the origin and reverses course along ℓ back to Q_0. Let γ_2 be the arc that travels along ℓ towards ∞ encircles all the finite branch points by a large circle about the origin and then reverses course along ℓ back to Q_0.*

Let $\mathcal{G} = (\gamma_2, \gamma_{1,1}, \ldots, \gamma_{p,1}, \ldots, \gamma_{1,k}, \ldots, \gamma_{p,k}, \gamma_1)$. By construction the paths can be jiggled slightly so that the conditions of Construction 2.2 are satisfied. Denoting $\left(\prod_{i=1}^{p}\gamma_{i,j}\right)$ by Γ_j we have,

$$\gamma_2 \prod_{j=1}^{k}\left(\prod_{i=1}^{p}\gamma_{i,j}\right)\gamma_1 = \gamma_2\left(\prod_{j=1}^{k}\Gamma_j\right)\gamma_1 = 1.$$

The inside product Γ_j is an ordered product over the branch points in a wedge. See Figure 2.

Proposition 3.5. *Let all notation be as in Construction 3.4 and let*

$$\mathcal{V} = (c_2, c_{1,1}, \ldots, c_{p,1}, \ldots, c_{1,k}, \ldots, c_{p,k}, c_1)$$

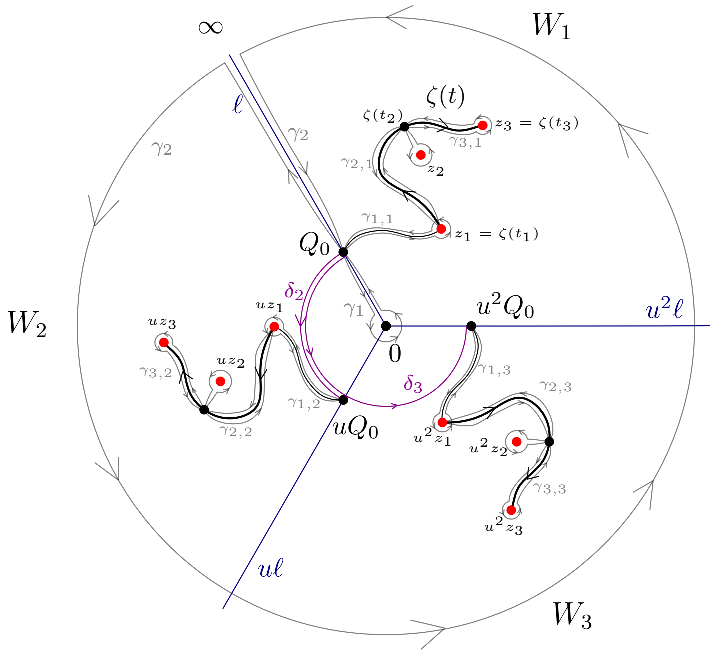

FIGURE 2. Construction 3.4.

be the corresponding generating vector. Then

$$\begin{aligned}
\delta_1\overline{h}_*(\gamma_1)\delta_1^{-1} &= \gamma_1 \\
\delta_1\overline{h}_*(\gamma_2)\delta_1^{-1} &= \Gamma_1^{-1}\gamma_2\Gamma_1 \\
\delta_1\overline{h}_*(\gamma_{i,j})\delta_1^{-1} &= \gamma_{i,j+1},\ 1 \le i \le p,\ 1 \le j \le k-1 \\
\delta_1\overline{h}_*(\gamma_{i,k})\delta_1^{-1} &= \gamma_1\gamma_{i,1}\gamma_1^{-1},\ 1 \le i \le p.
\end{aligned}$$

Letting $C_i = \left(\prod_{i=1}^{p} c_{i,j}\right)$*,then the* G *action extends to an action* $\widetilde{G}$ *on* S *with* $G \hookrightarrow \widetilde{G} \twoheadrightarrow \langle \overline{h} \rangle$ *if and only if there is an automorphism* ω *of* G *such that*

$$\begin{aligned}
\omega(c_1) &= c_1 \\
\omega(c_2) &= C_1^{-1}c_2C_1 \\
\omega(c_{i,j}) &= c_{i.j+1}, 1 \le i \le p,\ 1 \le j \le k-1 \\
\omega(c_{i,k}) &= c_1c_{i,1}c_1^{-1}, 1 \le i \le p.
\end{aligned}$$

Moreover

$$\widetilde{G} = \left\langle h, G : h^k \in G, hgh^{-1} = \omega(g), g \in G \right\rangle.$$

Proof. We leave to the reader the proofs of the first, third, and fourth formulas for the transforms of the elements of $\mathcal{G}$. For the second formula we write

$$\gamma_2\Gamma_1 \cdots \Gamma_k\gamma_1 = 1$$

and denoting $\gamma \to \gamma'$ the transform $\gamma' = \delta_1 \overline{h}_*(\gamma)\delta_1^{-1}$ we see that $\Gamma'_j = \Gamma_{j+1}$ for $1 \leq j \leq k-1$, and $\Gamma'_k = \gamma_1\Gamma_1\gamma_1^{-1}$. It follows that

$$\begin{aligned}
\gamma'_2\Gamma'_1 \cdots \Gamma'_k\gamma'_1 &= 1 \\
\gamma'_2\Gamma_2 \cdots \Gamma_k\gamma_1\Gamma_1\gamma_1^{-1}\gamma_1 &= 1 \\
\gamma'_2\Gamma_2 \cdots \Gamma_k\gamma_1\Gamma_1 &= 1 \\
\gamma'_2\Gamma_2 \cdots \Gamma_k\Gamma_1\Gamma_1^{-1}\gamma_1\Gamma_1 &= 1.
\end{aligned}$$

Now

$$\begin{aligned}
\Gamma_1 \cdots \Gamma_k &= \gamma_2^{-1}\gamma_1^{-1} \\
\Gamma_2 \cdots \Gamma_k\Gamma_1 &= \Gamma_1^{-1}\gamma_2^{-1}\gamma_1^{-1}\Gamma_1,
\end{aligned}$$

and

$$\begin{aligned}
1 &= \gamma'_2\Gamma_2 \cdots \Gamma_k\Gamma_1\Gamma_1^{-1}\gamma_1\Gamma_1 \\
&= \gamma'_2\Gamma_1^{-1}\gamma_2^{-1}\gamma_1^{-1}\Gamma_1\Gamma_1^{-1}\gamma_1\Gamma_1 \\
&= \gamma'_2\Gamma_1^{-1}\gamma_2^{-1}\Gamma_1.
\end{aligned}$$

It follows that

$$\gamma'_2 = \Gamma_1^{-1}\gamma_2\Gamma_1.$$

The rest of the proof is a straightforward application of Proposition 2.6. □

Example 3.2. Let G be the cyclic group of order 7 with generator x. From Example 3.1, we know there is a possible C_3 extension where N has signature $(3, 3, 7)$. Letting $B_G = \{1, u, u^2\}$ where u is a third root of unity, generating vectors from Proposition 3.5 will be of the form $(1, x^a, x^b, x^c, 1)$ where $a + b + c$ is divisible by 7 (note: c_1 and c_2 are trivial since neither 0 nor ∞ are in B_G, so the corresponding loops are trivial in the fundamental group). One such generating vector is $(1, x, x^2, x^4, 1)$. For this generating vector, it is easy to check that $\omega(x) = x^2$ is an automorphism of G which satisfies the given properties in Proposition 3.5 for extension. Thus $N = \langle x, h : hxh^{-1} = x^2\rangle$ is an extension of G.

Extending actions for more general groups

To determine other possible normal extensions by other groups we proceed as follows. For each possible K, we first find a representative of K so that $\{Q_1, \ldots, Q_t\}$ is a union of complete orbits of K. Then:

(1) For a given K find a set generators of K.
(2) For each generator $\overline{h}$ of a generating set for K carry out the analysis for a single automorphism to see if $\overline{h}$ lifts.

3.2. **The non-normal extension case.**

3.2.1. *Finding Possible Signatures for A.* Finding the possible signatures for A is more difficult than for N, so rather than provide an explicit statement, we describe the basic process.

The first step in this process is finding the possible indices of N in A. Now, since we are assuming A is an automorphism group of a compact Riemann surface of genus σ, there are natural bounds on the size of A, with maximal values arising when the signature for A has just three periods. For example, Table 2 from Lemma 3.2 in [21] gives all possible signatures for A when $|A| \geq \frac{13}{2}(2\sigma - 2)$.

Signature	Additional Conditions	$\lvert A\rvert$
$(3,3,n)$	$4 \leq n \leq 5$	$\frac{3n}{n-3}(2\sigma-2)$
$(2,5,5)$		$10(2\sigma-2)$
$(2,4,n)$	$4 \leq n \leq 10$	$\frac{4n}{n-4}(2\sigma-2)$
$(2,3,n)$	$7 \leq n \leq 78$	$\frac{6n}{n-6}(2\sigma-2)$

Table 2. Signatures for Large Automorphism Groups

Using these bounds, we get corresponding bounds on d, the index of N in A. Specifically, either

$$d = \frac{|A|}{|N|} \leq \frac{13}{2} \cdot \left(\sum_{i=1}^{s} \left(1 - \frac{1}{m_i}\right) - 2 \right), \tag{39}$$

where $(m_1, \dots, m_s)$ is the signature of N or the signature of A appears in Table 2 and the index d can be calculated exactly.

Next, if A does not have signature from Table 2, we can build the possible signatures for A as follows. Let $t_1, \dots t_o$ denote the orders of non-trivial elements of N. For a given index d which satisfies the inequality in equation (39), letting $d_1, \dots, d_q$ denote the divisors of d (including 1), A will have a signature of the form $((t_1 d_1)^{a_{1,1}}, (t_2 d_1)^{a_{2,1}}, \dots, (t_o d_q)^{a_{o,q}})$ where:

- the signatures for N and A and the index d satisfy equation (22)
- for each m_i, there exists an n_j with $m_i | n_j$
- the signatures for N and A are compatible with some permutation representation Φ given in part (1) of Theorem 2.3.

Remark 3.3. We do not need to build the explicit representation given in the last step – just know that a compatible representation exists.

Remark 3.4. The process we have described for building signatures for A can be streamlined significantly, especially when we know the specific structure of N and its corresponding signature.

We illustrate with an example.

Example 3.3. Starting with the group action with signature $(3, 3, 7)$ from Example 3.2, we find the possible signatures for non-normal extensions. First, equation (39) yields

$$d \leq \frac{13}{2} \cdot \frac{4}{21} < 2,$$

which is impossible, and so any signatures for non-normal extensions must come from Table 2. Since there must be periods divisible by both 3 and 7, this just leaves signatures of the form $(2, 3, n)$ where n is divisible by 7. Calculation shows that the only possible one of these signatures which satisfy equation (22) is $(2, 3, 7)$, so in particular, this is the only possible signature for a non-normal extension.

Remark 3.5. We note that the inclusion of signatures in Example 3.3 is already well known. Our purpose however was to illustrate the basic process of finding signatures for non-normal extensions.

3.2.2. *Primitive trivial core extensions.* Now suppose we have a non-normal extension $G < H$. We are going to focus on Case 3 described at the beginning of this section. We may find all H, generating vectors $\mathcal{V}_H$, the corresponding $S/G \to S/H$, and lifted branch sets, lifted generating sets $\mathcal{G}_G$, and generating vectors $\mathcal{V}_C$ in the steps below. Once the candidates have been found they need to be compared to the original B_G, $\mathcal{G}$, and $\mathcal{V}$.

Steps to find H:

1. Find the possible indices $d = |H| / |G|$ using the bound in (37).
2. For each d, search for primitive groups H of degree d whose point stabilizer is isomorphic to G.
3. For each H so determined, find all n-gonal signatures $\mathcal{S}_H$ such that an H-action with the given signature produces an n-gonal surface S with the given genus σ. Use the Riemann Hurwitz Theorem.

Steps to find signatures and generating vectors:

4. Using Theorem **2.3,** and the permutation representation of H on H/G find out which signatures $\mathcal{S}_H$ induce an n-gonal action of G with signature $\mathcal{S}$. Generating vectors are not needed at this stage, just the conjugacy classes of the elements of a generating vector.
5. For each signature found in Step 4 find all generating vectors $\mathcal{V}_H$ of H with the given signature.

Lifting Steps:

6. For each generating vector in Step 5 determine the map $S/G \to S/H$ as a rational function.
7. For each generating vector in Step 5 determine a lifted generating set $\mathcal{G}_H$.
8. For each map in Step 6 lift B_H to a branch set B on S/G.
9. For each generating vector $\mathcal{V}_H$ find a generating vector $\mathcal{V}_G$ of the G action (details below).

Comparison steps:

10. For each lift B in Step 7 compare B_H to B_G.
11. Compare the generating vector $\mathcal{V}_G$ with the original $\mathcal{V}$ (details below)

In the rest of the section we illustrate the steps above through example.

Finding H

Example 3.4. We start by considering the smallest non-Abelian example, $G = \Sigma_3$. Then G acts on a surface of genus 8 with signature $(2, 2, 2, 2, 2, 2)$ and generating vector $((1, 2), (1, 2), (2, 3), (2, 3), (1, 3), (1, 3))$. From equation (37) we get $d \leq 42 \times 1$. Here is a table of possible extensions computed using MAGMA.

H	t	$\|H/G\|$	potential $\mathcal{S}_H$	$\#\ \mathcal{V}_H/\|\mathrm{Aut}(H)\|$
Σ_4	3	4	$(4, 4, 4)$	0
Σ_4	4	4	$(2, 2, 2, 4)$	4
A_5	3	10	$(2, 5, 5)$	1

We see from the third column that there are two possible extensions. In the second row there are generically 4 different surfaces though for certain configurations some of the surfaces may be conformally equivalent.

Example 3.5. We consider the smallest simple example $G = A_5$. Using the primitive groups database in MAGMA we can check which primitive groups H have A_5 as a point stabilizer. There are 11 such groups H with primitive permutation degree less than 250. Among the groups, we have A_5, $A_5 \times A_5$, $SL(2, 11)$, $PSL(2, q)$ for $q = 16, 19, 29, 31$, and $A_5 \ltimes \mathbb{F}_q^r$ for $(q, r) = (2, 4), (3, 4)$, and $(5, 3)$.

Finding a $\mathcal{G}$ and $\mathcal{V}$

Let $U = S/H$, $B_H = \{R_1, \dots R_r\}$ and $\mathcal{H} = \{\delta_1, \dots, \delta_r\}$ be a generating set for $\pi_1(U^\circ, R_0)$, with Q_0 lying over R_0, and $(d_1, \dots, d_r)$ a generating vector for the H action. By covering space theory it may be shown that there are words $\psi_j \in \pi_1(U^\circ, R_0)$ such that

$$\gamma_j = \psi_j \left(\delta_{\zeta(j)}\right)^{e_j} \psi_j^{-1} \tag{40}$$

where $\pi_{H/G}(Q_j) = R_{\zeta(j)}$ and $e_j = o(d_{\zeta(j)})/o(c_j)$. Once we have $\mathcal{H}$ then we can compute an induced generating vector from a generating vector $\mathcal{V}_H = (d_1, \dots, d_r)$ via:

$$c_j = w_j \left(d_{\zeta(j)}\right)^{e_j} w_j^{-1}. \tag{41}$$

One way to compute the words in (40) is to have an explicit geometric model for $\pi_{H/G} : S/G \to S/H$ and then compute the images $\pi_{H/G}^*$ directly. This can be done for small examples.

Example 3.6. Let $G = \Sigma_3$, $H = \Sigma_4, \mathcal{V}_H = ((1, 2), (2, 3), (3, 4), (1, 2, 3, 4))$. If we let $R_4 = \infty$ the $\pi_{H/G}$ is a polynomial and a plausible map for $\pi_{H/G}$ is

$$\pi_{H/G} : z \to z^2 \left(3z^2 - 4(\lambda + 1)z + 6\lambda\right),$$

where λ is a parameter. In the domain of $\pi_{H/G}$ there is a ramification point of order 4 at ∞, and ramification point of order 2 at 0. The other ramification points are the other zeros of the derivative $\pi'_{H/G}(z) = 12z\,(z - 1)\,(z - \lambda)$, namely 1 and λ. The images of $0, 1, \lambda$ and ∞ under $\pi_{H/G}$ are $0, 2\lambda - 1, \lambda^3\,(2 - \lambda)$, and ∞, respectively. Certain values of λ must be excluded to keep the values distinct. The preimages as formulae in λ could be computed but the solutions are ungainly. For $\lambda = 3$ we get:

$$\begin{aligned}
\pi_{H/G}^{-1}(0) &= \left\{0, 0, \frac{8}{3} + \frac{1}{3}\sqrt{10}, \frac{8}{3} - \frac{1}{3}\sqrt{10}\right\}, \\
\pi_{H/G}^{-1}(5) &= \left\{1, 1, \frac{5}{3} + \frac{2}{3}\sqrt{10}, \frac{5}{3} - \frac{2}{3}\sqrt{10}\right\}, \\
\pi_{H/G}^{-1}(-27) &= \left\{3, 3, -\frac{1}{3} + \frac{2}{3}i\sqrt{2}, -\frac{1}{3} - \frac{2}{3}i\sqrt{2}\right\}, \\
\pi_{H/G}^{-1}(\infty) &= \{\infty, \infty, \infty, \infty\}.
\end{aligned}$$

Repeated entries indicate a ramification point.

Example 3.7. Let G and H be as in the example above, let $\mathcal{H} = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ be a generating system for the H action. The monodromy vector for the action of H is the same as the generating vector. Using only the information in the monodromy vector, one can draw a lift of the system $\mathcal{H}$ in S/H to S/G via $\pi_{H/G} : S/G \to S/H$ with appropriate punctures. The lift is a system of arcs and loops in S/G. One can select loops $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$ that encircle $\{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$ in some order. The $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$ can be modified by braid operations to achieve the correct ordering on B_G. For the sake of argument we are going to assume that no reordering is necessary.

$$\gamma_1 = \delta_4^{-1}\delta_1\delta_4,\ \gamma_2 = \delta_4^{-1}\delta_2\delta_4,\ \gamma_3 = \delta_1\delta_2\delta_3\delta_2^{-1}\delta_1^{-1}$$
$$\gamma_4 = \delta_1\delta_3\delta_1^{-1},\ \gamma_5 = \delta_2,\ \gamma_6 = \delta_3$$

and

$$c_1 = d_4^{-1}d_1d_4,\ c_2 = d_4^{-1}d_2d_4,\ c_3 = d_1d_2d_2^{-1}d_1^{-1}$$
$$c_4 = d_1d_3d_1^{-1},\ c_5 = d_2,\ c_6 = d_3.$$

We compute:

$$c_1 = (2,3), c_2 = (3,4), c_3 = (2,3)$$
$$c_4 = (3,4), c_5 = (2,3), c_6 = (3,4).$$

The group generated by the c_j is the symmetric group on $\{2,3,4\}$, the stabilizer of 1. To compare the generating vector with the original, we first conjugate the stabilizer of 1 to Σ_3 and then use the braid action.

Remark 3.6. In general the map $\pi_1(T^\circ, Q_0) \to \pi_1(U^\circ, R_0)$ can be computed directly from the monodromy vector $(\Phi(d_1), \ldots, \Phi(d_r))$, where $\Phi : H \to \Sigma_d$ is the monodromy representation the cosets of H/G.

4. STRONG BRANCHING AND WEAKLY MALNORMAL ACTIONS

In this section, we introduce the main ideas behind strong branching. We also introduce an additional condition which, when combined with strong branching, ensures the n-gonal subgroup is normal in the full automorphism group.

4.1. **Strong branching.** In [1], Accola introduced strong branching.

Definition 4.1. Let $\pi : S_1 \to S_2$ be a branched covering of degree n. The covering π is *strongly branched* if

(42) $$R_\pi > 2n(n-1)(\sigma_2 + 1),$$

or, equivalently,

(43) $$\sigma_1 > n^2\sigma_2 + (n-1)^2.$$

If the conditions do not hold then π is a called *weakly branched.*

Remark 4.1. If S_2 has genus 0 then the formulas become

(44) $$R_\pi > 2n(n-1)$$

(45) $$\sigma_1 > (n-1)^2.$$

For conciseness, if the map $S \to S/G$ is strongly branched, we shall also say that the group action of G is strongly branched.

In the context of finding automorphism groups, as indicated in Section 3, for a given n-gonal group, finding a normal extension (if one exists) is a difficult but tractable problem. In contrast however, finding non-normal extensions seems much more difficult. Strong branching ensures the existence of certain normal subgroups in the full automorphism group of a surface, thus making calculation of A more straightforward. Specifically, we have:

Proposition 4.1. *Let G be a group of automorphisms acting on a surface S such that $S \to S/G$ is strongly branched. Then there is a unique, normal, non-trivial subgroup M of $Aut(S)$ such that $M \leq G$, and $S \to S/M$ is strongly branched.*

Proof. By the proof of Corollary 3 of [1] there is a unique, maximal intermediate surface $S \to U \to S/G$ such that $S \to U$ is a Galois, strongly branched covering of degree exceeding one. Accordingly, there is a non-trivial subgroup $M \leq G$ such that $U = S/M$, and, as U is unique, M must be normal. □

4.2. **Number of branch points and families of actions.** Next we focus on the number of branch points for each action.

Number of branch points

One unfortunate drawback of using strong branching is that either the cut-off genus in equation (45) tends to be large or the number of branch points in the quotient is large. When considering surfaces as regular, branched coverings of quotients with branch points, it is more natural to use the number and order of the branch points, action signatures, and group orders as constraints rather than the genus of S, as in equation (45). Specifically, assuming a regular n-gonal action $\pi : S \to S/G$, and using equation (6), the strong branching criterion (44) can be written

$$\sum_{j=1}^{t} \left(1 - \frac{1}{n_j}\right) > 2(n-1), \tag{46}$$

upon noting that $\left|\pi^{-1}(Q_j)\right| n = n/n_j$.

If all the $n_j = n$, as in the prime cyclic case and the superelliptic case then we must have:

$$t > 2n. \tag{47}$$

The worst possible case (largest t) is when $n_j = 2$ for all j and then we must have $t > 4(n-1)$. For a weaker lower bound on t, if we replace all the $1 - \frac{1}{n_j}$ by 1 we must have

$$t > 2(n-1). \tag{48}$$

We can use equation (46) to estimate the number of weakly branched, potential signatures for a G action. Let $e_1, \ldots, e_r$ be the orders of non-trivial elements of G, In a given n-gonal signature $(n_1, \ldots, n_t)$, let $x_k = |\{j : n_j = e_k\}|$, then from equation (46) a signature is weakly branched if

$$\sum_{k=1}^{r} \left(1 - \frac{1}{e_k}\right) x_k \leq 2(n-1). \tag{49}$$

The number of nonnegative integer solutions to this equation has a reasonable approximation (lower bound) by the volume of the simplex in the positive $\mathbb{R}^r$orthant bounded by the hyperplane $\sum_{k=1}^{r} \frac{e_k - 1}{e_k} x_k = 2(n-1)$. Thus

$$\#signatures \geq \frac{(2n-2)^r}{r!} \prod_{k=1}^{r} \frac{e_k}{e_k - 1}.$$

For $G = A_5$ the smallest non-Abelian simple group, $e_1 = 2, e_2 = 3, e_3 = 5$ and

$$\#signatures \geq \frac{118^3}{3!}\frac{2}{1}\frac{3}{2}\frac{5}{4} = 1026895.$$

The actual number of potential n-gonal signatures is 1053238, directly computed using MAGMA.

The proportions of strongly branched actions and n-gonal actions

One obvious lingering question underlying our work is how frequently strong branching can be used in determining full automorphism groups. Specifically, our goal is to develop methods to find full automorphism groups for n-gonal surfaces and for surfaces which admit strongly branched group actions (and combinations of both). In this context, the question of how frequently these methods can be used for a fixed genus comes down to what proportion of group-signature pairs are either strongly branched or n-gonal for a fixed genus. The large bound on the number of branch points and/or quotient genus suggests that group-signature pairs with groups that are strongly branched are rare. In contrast however, the high frequency of genus-0 actions suggests that group-signature pairs for automorphism groups of n-gonal surfaces should actually be quite frequent. It is not immediately clear how to prove such assertions in general, but the available data for low genus actions (such as Breuer's lists, up to genus 48, in [5]) supports the following:

- In a fixed genus, the proportion of total actions which are strongly branched lies roughly between 2% and 5%.
- The number of group-signature pairs for n-gonal actions is a substantial proportion of all actions. Indeed, over all genera less that 49, 55% of actions are n-gonal.
- In a fixed genus, the proportion of n-gonal actions which are strongly branched is around 10% on average.

In particular, the frequency with which n-gonal surfaces seem to occur certainly supports further development of techniques to find their automorphism groups. Though strong branching occurs less frequently, further study of strongly branched actions makes sense since they are more tractable than the general case and the strong branching condition provides a good theoretical cut-off point.

Remark 4.2. According to equations (46), (47), and (48), in the presence of strong branching, we have a large dimensional family $\pi : E \to B$ where the typical fiber has a G action with a given signature $(n_1, \ldots, n_t)$, e.g., the cyclic n-gonal families. In many cases, for a typical fiber $\pi^{-1}(b)$ the action of G on $\pi^{-1}(b)$ constitutes the full automorphism group A, and strong branching does not tell us anything special since the guaranteed normal subgroup satisfies $M = G = A$. In this case we have a large open set $B^\circ \subset B$ of G-equisymmetry. For special values of b (actually subvarieties) $\mathrm{Aut}(\pi^{-1}(b))$ is strictly larger than the image of G. The various possibilities for $M \leq G \leq A$ with $M \lhd A$ correspond to subvarieties of B, with equisymmetric actions of A.

4.3. **Weakly malnormal actions.** We now introduce a key concept that allows us to use strong branching to guarantee normality. Recall that a subgroup G of a group A is called malnormal if $G \cap xGx^{-1}$is trivial whenever $x \notin G$. We generalize this definition.

Definition 4.2. A subgroup G of a group A is said to be *weakly malnormal* if and only if $G \cap xGx^{-1}$ is trivial when $x \notin N_A(G)$. Now suppose that G acts on S via $\epsilon : G \hookrightarrow \mathrm{Aut}(S)$. We say that the action of G on S is weakly malnormal if $\epsilon(G)$ is a weakly malnormal subgroup of A, the full automorphism group S.

Next, we present some useful facts about weak malnormality.

Proposition 4.2. *Let G act on S and $A = \mathrm{Aut}(S)$. The following statements characterize weakly malnormal actions:*

(1) *If $G \trianglelefteq A$, then G is automatically weakly malnormal in A.*
(2) *If G is weakly malnormal in A, but not normal, then G has a trivial core in A.*
(3) *If the subgroup $G < A$ is weakly malnormal then for any non-trivial $M \leq G$, we must have $N_A(M) \leq N_A(G)$.*
(4) *If G is cyclic then $N_A(G) = N_A(M)$ for any non-trivial $M \leq G$ if and only if G is weakly malnormal in A.*

Proof. Statements 1 and 2 are left to the reader. To prove statement 3, suppose $x \in N_A(M)$. Then for all $x \in A$, $G \cap xGx^{-1} \geq M \cap xMx^{-1} = M > \{1\}$. Since G is weakly malnormal, it follows that $x \in N_A(G)$. For statement 4, suppose that G is cyclic and weakly malnormal in A. We already have $N_A(M) \leq N_A(G)$. Suppose that $x \in N_A(G)$. Then M and xMx^{-1} both lie in G and so must equal each other since G has a unique subgroup of order $|M|$. It follows that $x \in N_A(M)$ and $N_A(G) = N_A(M)$. For the converse that $x \in A - N$ and suppose that $M = G \cap xGx^{-1}$ is not trivial, then both $M \leq G$ and $x^{-1}Mx \leq G$, so that M and $x^{-1}Mx$ both equal the unique subgroup of G of order $|M|$. Thus $x \in N_A(M) = N_A(G)$. This contradicts $x \in A - N$ so that we must have $G \cap xGx^{-1} = \{1\}$. □

In the next Proposition we see how to use strong branching and weak normality to prove normality results.

Proposition 4.3. *Suppose that G has a weakly malnormal action on S and that $\pi_G : S \to S/G$ is strongly branched. Then G is normal in A.*

Proof. Let M be the non-trivial normal subgroup of A contained in G guaranteed by Proposition 4.1. If G is not normal then

$$M = \bigcap_{x \in A} xMx^{-1} \leq \bigcap_{x \in A} xGx^{-1} = \{1\},$$

a contradiction. □

According to Proposition 4.1 if $S \to S/G$ is strongly branched then $\mathrm{Core}_A(G)$ is not trivial since G is guaranteed to have a non-trivial normal subgroup. In the introduction it was suggested that if $M = \mathrm{Core}_A(G) < G$ then we look at actions on S/M. Some useful properties of such actions are summarized in the next proposition.

Proposition 4.4. *Let S is a Riemann surface, $A = \mathrm{Aut}(S)$ and $G \leq A$ and let $M = \mathrm{Core}_A(G)$ be a proper subgroup of G. Then $\overline{S} = S/M$ is a surface upon which both $\overline{A} = A/M$ and $\overline{G} = G/M$ act naturally, and $\overline{A} \leq \mathrm{Aut}(\overline{S})$. Moreover, if $G \lneq A$ then $\overline{S} \to \overline{S}/\overline{G}$ is not strongly branched.*

Proof. The proof that $\overline{S}$ is a Riemann surface and that $\overline{A}$ and $\overline{G}$ act naturally is straightforward.

To show that if $G \lneq A$, then $\overline{S} \to \overline{S}/\overline{G}$ is not strongly branched, we proceed by contradiction. If $\overline{S} \to \overline{S}/\overline{G}$ is strongly branched then there would be a non-trivial subgroup of $\overline{G}$ that is normal in $\mathrm{Aut}(\overline{S})$. However, if $G \lneq A$, then

$$\mathrm{Core}_{\overline{A}}(\overline{G}) = \bigcap_{\overline{x}\in\overline{A}} \overline{x}\,\overline{G}\,\overline{x}^{-1} = \bigcap_{x\in A} \left(xGx^{-1}/M\right) = M/M = \{1\}$$

and

$$\mathrm{Core}_{\mathrm{Aut}(\overline{S})}(\overline{G}) \leq \mathrm{Core}_{\overline{A}}(\overline{G}) = \{1\}\,.$$

Therefore G does not contain a non-trivial subgroup that is normal in $\mathrm{Aut}(\overline{S})$, a contradiction. □

5. Determining Automorphism Groups

We finish by illustrating the tools and techniques we have developed to determine full automorphism groups of families of surfaces through explicit examples. We shall start with the most well known family – cyclic n-gonal surfaces – providing a brief survey of the known results, and introducing new ones. Following this, we shall provide a general outline of how to use strong branching in determining full automorphism groups when there is an n-gonal group which is simple, and then illustrate by exploring in detail the family of surfaces with n-gonal group isomorphic to the alternating group A_5. Throughout the whole section, we provide explicit details of how the techniques we employ can be used or adapted to other similar families. Where we feel confident, we will also provide conjectures that we hope will motivate further work.

5.1. **Cyclic n-gonal actions.**

A ubiquitous and important case of group actions are those for which G is cyclic and S/G has genus 0. Such surfaces have tractable equations. A convenient form for such surfaces is given in the following.

Example 5.1. Let $m_1, \ldots, m_t$, and n be integers satisfying:

(1) $1 \leq m_j < n$,
(2) n divides $m_1 + \cdots + m_t$, and
(3) $\gcd(m_1, \ldots, m_t) = 1$.

Then the surface $\overline{S}$ defined by

$$y^n = (x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_t)^{m_t}, \tag{50}$$

where the $a_1, \ldots, a_t$, are distinct, is an irreducible cyclic n-gonal surface. If $m_j > 1$ the point $(a_j, 0)$ is singular. There are $d_j = \gcd(m_j, n)$ local branches of $\overline{S}$ at $(a_j, 0)$. The normalization map $\nu : S \to \overline{S}$ resolves the singularities and d_j points lie over $(a_j, 0)$. The action of $G = C_n$ on $\overline{S}$ is defined by $(x, y) \to (x, u^k y)$ where $u = \exp(2\pi i/n)$. This action lifts to S and the quotient map $\pi_G : S \to S/G$, called the n-gonal morphism, is given by $\pi_G : S \xrightarrow{\upsilon} \overline{S} \xrightarrow{\pi} \mathbb{P}^1$ where $\pi(x, y) = x$. The map π_G is branched over each $Q_j = a_j$, but is unbranched at ∞, by condition 2. Letting g be the automorphism $(x, y) \to (x, uy)$, we have $c_j = g^{m_j}$ and c_j fixes the d_j points lying over Q_j. The order of c_j is n/d_j so $n = n_j d_j$. For more details see [7].

Remark 5.1. If $n = p$ is prime we call the surface p-gonal.

Example 5.2. Two interesting special cases of cyclic n-gonal surfaces are *superelliptic surfaces* and *generalized superelliptic surfaces*. Superelliptic surfaces are those surfaces of the form

$$y^n = f(x)$$

where $f(x)$ is square free and n does not divide the degree of f. The point at ∞ will be a point of ramification. Of special interest is the case $n = p$ a prime. A *generalized superelliptic surface* has an equation as given in (50) where $\gcd(m_j, n) = 1$, or alternatively those cyclic n-gonal surfaces whose cyclic group of automorphisms has signature $(n, \ldots, n)$.

Example 5.3. Continuing Example 5.1, consider the family of curves defined by

$$y^n = (x - a_1)^{m_1} \cdot \cdots \cdot (x - a_t)^{m_t} \tag{51}$$

with $(a_1, \ldots, a_t) \in \mathbb{C}^t - \Delta$, where Δ is the multidiagonal. The family is constructed by first taking all points of the form $(x, y, a_1, \ldots, a_t) \in \mathbb{C}^{t+2}$ that satisfy (51) and then forming the closure E_1 of these points in $\mathbb{P}^2 \times (\mathbb{C}^t - \Delta)$. After normalizing E_1 we get $\pi : E \to B = \mathbb{C}^t - \Delta$ such that $\pi(x, y, a_1, \ldots, a_t) = (a_1, \ldots, a_t)$. The action ϵ_b, $b \in B$ of $G = C_n$ on E_1 is defined by $(x, y) \to (x, u^k y)$ where $u = \exp(2\pi i/n)$. The action is then lifted to E.

Remark 5.2. Every n-gonal action of a group G branched over t points can be included in a family $\pi : E \to B$ where B is a finite covering of $\mathbb{C}^t - \Delta$ (Hurwitz space).

5.1.1. *Determining automorphism groups of cyclic p-gonal surfaces.* Though full results are known, see for example [22], we briefly describe how to determine the full automorphism group when G has prime order p. In this case, the strong branching cut-off is $\sigma = (p - 1)^2$ and so we have:

Proposition 5.1. *For prime $|G|$, if $\sigma > (p - 1)^2$ then G is normal in A.*

As outlined in Section 1, we split up the classification of automorphism groups into the two cases of whether or not G is normal.

The normal case

Assuming that G is normal, then $N = A$ satisfies the short exact sequence

$$G \hookrightarrow N \twoheadrightarrow K$$

Determining the possible solutions for N is straightforward, with most cases being split extensions. Next, for each possible N, we can use Proposition 3.3 to construct possible signatures for N and then determine whether or not such an action exists by constructing generating vectors, or showing none exist. We illustrate with an example.

Example 5.4. When $K = C_k$, the solutions to the short exact sequence

$$G \hookrightarrow N \twoheadrightarrow C_k.$$

are a direct product $G \times C_k$, a semi-direct product $G \rtimes C_k$ and the cyclic group C_{kp} (note that for certain k, these groups might coincide).

Since the signature of C_k is (k,k) and the signature of G is $(\underbrace{p,\ldots,p}_{r-times})$, using Proposition 3.3, the possible signatures of N are

$$(0;k,k,\underbrace{p,\ldots,p}_{r/k-\text{times}}),(0k,kp,\underbrace{p,\ldots,p}_{(r-1)/k-\text{times}}),(0;kp,kp,\underbrace{p,\ldots,p}_{(r-2)/k-\text{times}}).$$

When a non-trivial semi-direct product $G \rtimes C_k$ exists, the only possible signature for which there can exist a generating vector is $(0;k,k,p,\ldots,p)$.

If $\gcd(p,k)=1$, then $C_k \times C_p = C_{kp}$, and any of the three signatures could act as the signature of such a group action.

Finally, if $\gcd(p,k)=p$, then $C_k \times C_p$ and C_{kp} are distinct. In this case, when $A = C_k \times C_p$, a generating vector could only exist for the signature $(0;k,k,p,\ldots,p)$ and for $A = C_{kp}$, a generating vector could only exist for the signature $(0;kp,kp,p,\ldots,p)$.

In nearly all cases, a generating vector for the given group exists and is easy to construct. We leave the details to the reader.

See [8] for additional examples on normal extensions of cyclic actions.

The non-normal case

Now suppose that G is not normal in A. In this case, as expected, determining the possible A and signatures requires some ad hoc argumentation, so we refer to [22] for full details. We survey the basic steps here simplifying where possible.

We first note that automorphism groups for small primes can be found computationally using Breuer's database, [5]. For a given p, the strong branching cut-off is $\sigma = (p-1)^2$, and so each A for $p \leq 7$ can be determined. Using this database, we obtain four different automorphisms groups whose details we summarize in the first four rows of Table 3. We henceforth then assume that $p \geq 11$.

Next, using the strong branching cut-off and the Riemann-Hurwitz formula, it is easy to show that when $p \geq 11$, any Sylow subgroup S of A has order either p^2 or p. We analyze these two cases individually.

First suppose that S, a Sylow p-subgroup of A, has order p^2. If S is cyclic, it must have signature $(p^2,p^2,\underbrace{p,\ldots,p}_{\ell-\text{times}})$, see Example 5.4. For signatures of this form, the strong branching cut-off yields (p^2,p^2,p) as the only possibility. If S is elementary Abelian, then it must have signature $(\underbrace{p,\ldots,p}_{\ell-\text{times}})$, and again using the strong branching cut-off, we must have $\ell=3$ or $\ell=4$. Each of these signatures can now be analyzed individually, and by doing so we find three different families of surfaces with non-normal overgroup, see the last three rows in Table 3.

Now suppose that $p^2 \nmid |A|$. By Corollary 3.4 of [22], we know that $N > G$, and for the sake of simplicity, we also assume $K \neq C_k$. By looking at stabilizers of fixed points, we see that the signature of A differs only slightly from the signature of N, see Lemma 7.1 of [22]. Specifically:

Lemma 5.2. *There exists integers $m_1,\ldots,m_{\nu_1},o_1,\ldots,o_\tau$, o_i a multiple of p and each integer m_j and o_i/p relatively prime to p such that:*

(1) *the signature of N is $(m_1,\ldots,m_{\nu_1},o_1,\ldots,o_\tau,\underbrace{p,\ldots,p}_{\ell\ times})$,*

(2) *the signature of K is $(m_1,\ldots,m_{\nu_1},o_1/p,\ldots,o_\tau/p)$*

(3) *the signature of A is* $(n_1, \dots, n_{\nu_2}, o_1, \dots, o_\tau, \underbrace{p, \dots, p}_{\ell \text{ times}})$.

Moreover, each m_i must divide at least one n_j.

Next, instead of estimating the index d of N in A as outlined in Section 3.2, we can use Lemma 5.2 and equation (22) to calculate it explicitly:

$$d = \frac{-2 + \sum_{i=1}^{\nu_1} \left(1 - \frac{1}{m_i}\right) + \sum_{i=1}^{\tau} \left(1 - \frac{1}{o_i}\right) + l\left(\frac{p-1}{p}\right)}{-2 + \sum_{i=1}^{\nu_2} \left(1 - \frac{1}{n_i}\right) + \sum_{i=1}^{\tau} \left(1 - \frac{1}{o_i}\right) + l\left(\frac{p-1}{p}\right)} \tag{52}$$

which we can then simplify to:

$$d = 1 + \frac{\sum_{i=1}^{\nu_1} \left(1 - \frac{1}{m_1}\right) - \sum_{i=1}^{\nu_2} \left(1 - \frac{1}{n_i}\right)}{-2 + \sum_{i=1}^{\nu_2} \left(1 - \frac{1}{n_i}\right) + \sum_{i=1}^{\tau} \left(1 - \frac{1}{o_i}\right) + l\left(\frac{p-1}{p}\right)}. \tag{53}$$

Under the assumption that $p \geq 11$, we know all the possible signatures for K. Therefore it is straightforward, though time consuming, to show, except for a small number of cases which can be easily checked by hand, that if the extension is not normal, we must have $d < 12$. However, by Sylow theory, we know the index d of N in A has to be congruent to 1 modulo p, which is impossible since $p \geq 11$. Hence there are no further non-normal extensions of p-gonal groups to those already appearing in Table 3.

p	Signature of A	Signature of N	Genus	Group A
3	$(0; 2, 3, 8)$	$(0; 2, 2, 2, 3)$	2	$GL(2, 3)$
3	$(0; 2, 3, 12)$	$(0; 3, 4, 12)$	3	$[48, 33]$
5	$(0; 2, 4, 5)$	$(0; 4, 4, 5)$	4	S_5
7	$(0; 2, 3, 7)$	$(0; 3, 3, 7)$	3	$\mathrm{PSL}(2, 7)$
$p \geq 5$	$(0; 2, 3, 2p)$	$(0; 2, p, 2p)$	$\frac{(p-1)(p-2)}{2}$	$(C_p \times C_p) \rtimes S_3$
$p \geq 3$	$(0; 2, 2, 2, p)$	$(0; 2, 2, p, p)$	$(p-1)^2$	$(C_p \times C_p) \rtimes V_4$
$p \geq 3$	$(0; 2, 4, 2p)$	$(0; 2, 2p, 2p)$	$(p-1)^2$	$(C_p \times C_p) \rtimes D_4$

Table 3. Automorphism Groups of p-gonal Surfaces when $A \neq N$

5.1.2. *Strong branching and general cyclic n-gonal surfaces.* The obvious natural question to ask is whether the techniques we adopted for cyclic p-gonal surfaces can be used to determine full automorphism groups for other cyclic n-gonal surfaces. Strong branching played a key role in determining these groups as it ensured that there were only finitely many cases for which $A \neq N$, and from there we could apply ad hoc argumentation to construct the signature of A from the signature of N. Unfortunately the following example shows that for general cyclic n-gonal surfaces, strong branching does not ensure normality, and in particular, it is possible to construct infinitely many n-gonal surfaces for which $A \neq N$.

Example 5.5. Let $A = \langle x, y | x^4 = y^3 = xyx^{-1} = y^{-1} \rangle$, and $G = \langle x \rangle$. The group A has order 12 and G is a cyclic subgroup of order 4. We can define a generating vector for A with signature $(0; \underbrace{2, \ldots, 2}_{r \text{ times}}, 4, 4, 4, 4)$ for r even as follows:

$$(x^2, x^2, \ldots, x^2, x, x^{-1}, x, x^{-1}y^2)$$

Using Theorem 2.3, it is easy to show that the signature of the subgroup G is $(0; \underbrace{2, \ldots, 2}_{3r \text{ times}}, 4, 4, 4, 4)$ and the corresponding genus of the surface S on which A acts is $\sigma = 3r + 7$.

In the context of determining full automorphism groups, we observe that S is cyclic 4-gonal and the group G is never normal in A. However, the genus of S can be made arbitrarily large, so in particular, we can construct infinitely many cyclic 4-gonal surfaces of arbitrarily large genus for which a cyclic 4-gonal subgroup is not normal in A. A similar example can be constructed for $n = 9 = 3^2$, though the same construction fails for larger primes.

These group actions provide examples of strongly branched actions where the subgroup M from Proposition 4.1 is strictly contained in G.

5.1.3. *Generalized superelliptic surfaces.* The key result in determining automorphism groups for p-gonal surfaces was the fact that there were only finitely many group-signature pairs for which $A \neq N$, and this was due to strong branching – provided $\sigma > (p-1)^2$, G was guaranteed to be normal. In contrast, Example 5.5 showed there is little hope that strong branching will allow us to easily determine automorphism groups of all cyclic n-gonal surfaces. Therefore, this leads to the question of whether there are families of cyclic n-gonal surfaces, aside from the p-gonal ones, for which strong branching ensures normality of the cyclic n-gonal group in the full automorphism group. One such class is the generalized superelliptic surfaces (which includes the superelliptic surfaces).

Proposition 5.3. *Suppose that S is a generalized superelliptic n-gonal surface with cyclic automorphism group G. Further suppose that $S \to S/G$ is strongly branched, $\sigma > (n-1)^2$, $n = |G|$. Then G is normal in A.*

Proof. Since S is generalized superelliptic, then the stabilizer subgroup of G of any fixed point P is of order n, or equivalently $G_P = G$, if $G_P > \{1\}$. Let M be the normal subgroup of A contained in G, guaranteed by Proposition 4.1. Now suppose that P is any fixed point of G and that $x \in A - N$ satisfying $G \cap xGx^{-1} = M > \{1\}$. Then $G_P \geq M > \{1\}$ and so $G_P = G$. Next $G_{xP} = xG_Px^{-1} = xGx^{-1} \geq M > \{1\}$. It follows that $G_{xP} = G$ and $xGx^{-1} = G_{xP} = G$, a contradiction to $x \in A - N$. ☐

The importance of Proposition 5.3 is that G is normal when $\sigma > (n-1)^2$, and hence just like with the p-gonal case, for a given n, there are only finitely many possible A's for which $A \neq N$. Now, for a superelliptic surface S, when $A = N$, all possible A and the corresponding signatures were determined in [18]. In particular, the problem of complete classification comes down to analyzing just the A for which $A \neq N$.

To date, such a classification remains elusive. However, computational results for small n ($n \leq 12$), and attempts at generalizing the tools and techniques used for the cyclic p-gonal case suggest that there are no further families of groups, see

[9]. Consequently, we conjecture that the families already discovered (extended for all n) are the only possible ones for which $A \neq N$. Specifically:

Conjecture 5.4. *Suppose S is generalized superelliptic with $A \neq N$. Then A is one of the groups given in Table 3.*

See [15] for additional details on generalized superelliptic surfaces.

5.1.4. *Cyclic n-gonal cases which are not superelliptic.* Suppose now that $G = C_n$, and let S, A and N be as before. The strong branching condition only guarantees that there is cyclic subgroup $M = C_m \trianglelefteq A$ with $1 < m \leq n$. We would like to study cases where C_m is a proper subgroup of G, and to be specific we will focus on examples where $n = p^2$. The analysis using strong branching works as follows, assuming a classification of surfaces of any genus, with action group C_m.

(1) Assume $S \to S/G$ is strongly branched to obtain $M \trianglelefteq G$ with $\{1\} < M \trianglelefteq A$. We may assume that $M = \mathrm{Core}_A(G)$.
  (a) If $M = G$ then compute $A = N$ as an extension of G using the methods in Section 3.1.
  (b) If $G \ntrianglelefteq A$ then consider the quotient case $\overline{S} = S/M$, and the series of groups $\overline{G} \leq \overline{A} \leq \mathrm{Aut}(S')$ where $G' = G/M$, $A' = A/M$, $A'' = \mathrm{Aut}(S')$. Determine A' as a subgroup of A'' and then solve $M \hookrightarrow A \twoheadrightarrow A'$.

(2) If $S \to S/G$ is not strongly branched then use the methods of Section 3.2 to find A, assuming $A \neq N$. There are only finitely many cases to consider.

We will only consider what happens where $M < G$. Let us first consider the generalities of case $n = p^2$, and then work specific examples for low primes. To help with the bookkeeping of the numerous branch points we use the following notation. For $0 < k < n$ define

$$u_k = \left|\left\{j : c_j = x^k\right\}\right|.$$

A branch point has order p or p^2. If we let t_1 be the number of branch points of order p and t_2 be the number of branch points of order p^2, then we have:

$$\sum_{k=1}^{p^2-1} k u_k = 0 \bmod p^2,$$

$$t_1 = \sum_{k=1}^{p-1} u_{pk},$$

$$t_2 = t - t_1 \geq 2.$$

We need $t_2 \geq 2$, otherwise $|\langle c_1, \ldots, c_t \rangle| = p < |G|$. Thus

$$R_{\pi_G} = n \sum_{j=1}^{t} \left(1 - \frac{1}{n_j}\right) = p(p-1)t_1 + \left(p^2 - 1\right) t_2. \tag{54}$$

Using equation (14) the genus of S is given by

$$\begin{aligned}\sigma &= 1 + p^2(-1) + \frac{p^2}{2}\left(\frac{p-1}{p} t_1 + \frac{p^2-1}{p^2} t_2\right) \\ &= 1 - p^2 + \frac{p(p-1)}{2} t_1 + \frac{p^2-1}{2} t_2.\end{aligned}$$

Using equations (46) and (54), we see how many branch points are needed for strong branching:

$$\frac{p-1}{p}t_1 + \frac{p^2-1}{p^2}t_2 > 2(p^2-1)$$

or

$$t_1 > 2p(p+1) - \frac{p+1}{p}t_2. \tag{55}$$

Suppose $G = \langle x \rangle$, and assume the non-trivial subgroup guaranteed by strong branching is $M = \langle x^p \rangle$. According to Proposition 2.4 the number of ramification points of M acting on S is $pt_1 + t_2$ each with ramification order p and so $R_{\pi_M} = (pt_1 + t_2)(p-1)$. By equation (4) the genus σ' satisfies

$$\begin{aligned}
\sigma' &= 1 + \frac{1}{2p}\left(2(\sigma-1) - R_{\pi_M}\right) \\
&= 1 + \frac{1}{2p}\left(2\left(-p^2 + \frac{p(p-1)}{2}t_1 + \frac{p^2-1}{2}t_2\right) - (pt_1+t_2)(p-1)\right) \\
&= \frac{(p-1)}{2}t_2 + 1 - p.
\end{aligned}$$

The possible automorphism groups of S' are known from the classification of p-groups, except that extra work is needed for $\sigma' = 0, 1$. The automorphism group of S can be pieced together from $Aut(S')$ and M. If we assume that G is not normal in A then M is normal by the strong branching condition.

Example 5.6. Let us make a table of σ, σ' and the describe the cases for small primes $p = 2, 3, 5, 7$. According to Harvey, t_2 must be even when $p = 2$. Assuming strong branching we get

p	σ	σ'	restriction
2	$-3 + t_1 + 3\frac{t_2}{2}$	$\frac{t_2}{2} - 1$	$t_1 > 12 - 3\frac{t_2}{2}$
3	$-8 + 3t_1 + 4t_2$	$t_2 - 2$	$t_1 > 24 - \frac{4}{3}t_2$
5	$-24 + 10t_1 + 12t_2$	$2t_2 - 4$	$t_1 > 60 - \frac{6}{5}t_2$
7	$-48 + 21t_1 + 24t_2$	$3t_2 - 6$	$t_1 > 112 - \frac{8}{7}t_2$

Let us now describe examples of such possible groups.

Example 5.7. Let p and q be primes such that p divides $q - 1$. Write $C_{p^2} = \langle x \rangle$ and $C_q = \langle y \rangle$ in multiplicative format. Let $a \in C_q^* = \mathrm{Aut}(C_q)$ be such that $a^p = 1 \bmod q$ which exists by divisibility conditions. Let C_{p^2} act upon $C_q = \langle y \rangle$ by

$$\theta : C_{p^2} \to C_q^* = \mathrm{Aut}(C_q)$$

$$x^j \cdot y^k = \theta(x^j)(y^k) \to \left(y^k\right)^{a^j}.$$

Then the semi-direct product

$$A = C_{p^2} \ltimes C_q = \left\langle x, y : x^{p^2} = y^q = 1, x^{-1}yx = y^a \right\rangle$$

satisfies:

- $\langle y \rangle, \langle x^p \rangle \lhd A$,
- $x^p y$ has order pq, and
- $\langle x \rangle \ntriangleleft A$. Indeed any cyclic subgroup of order p^2 is self-normalizing.

Example 5.8. Let $A = C_9 \ltimes C_7 = \langle x, y : x^9 = y^7 = 1, x^{-1}yx = y^2 \rangle$. ($A$ is SmallGroup(63,1) in MAGMA). The vector $\mathcal{V} = (x^7, xy, xy^5)$ is a generating vector with signature $(9, 9, 9)$, yielding a surface of genus 22. Using MAGMA as in Remark 2.3 we see that there is a cyclic subgroup of order 9 whose signature is $(3, 3, 3, 3, 3, 3, 9, 9, 9)$. This action is not strongly branched and so the non-normal extension is not a surprise. Next consider a generating vector obtained from $\mathcal{V}$ prepending 3 copies of x^3 to $\mathcal{V}$, i.e., $(x^3, x^3, x^3, x^7, xy, xy^5)$. The signature of the action of G has signature $(3^{27}, 9^3)$ and the surface has genus 85. This action is strongly branched. It is conceivable that the automorphism group is larger but the subgroup M must be $\langle x^3 \rangle$. Also of interest, in this case $\sigma' = 1$ and so the quotient S/G is a torus that supports a group of automorphisms of the form $C_3 \ltimes C_7$.

5.2. **Simple n-gonal groups and strong branching.** In the current literature, the only families for which strong branching has been used to determine full automorphism groups are cyclic groups, but there are other families for which strong branching should provide the framework for determining all possible automorphism groups. The most obvious of these is the family of simple groups. Specifically, since simple groups have no normal subgroups, we have:

Proposition 5.5. *For G simple, if $\sigma > (|G| - 1)^2$ then G is normal in A.*

In particular, for a given simple n-gonal group G, Proposition 5.5 ensures that there are only finitely many possible A for which $A \neq N$ and so we can use the same techniques for finding A as we have previously outlined.

The normal case for simple groups

When $A = N$, so G is normal in its full automorphism group, the possible signatures for A satisfy Proposition 3.3 with the possible A being solutions to the short exact sequence:

$$G \hookrightarrow A \twoheadrightarrow K,$$

We note that for a given simple group G, there could be a tremendous number of solutions to this short exact sequence, Moreover, there is no guarantee that these solutions should all split as we saw with the cyclic p-gonal case. In particular, for an arbitrary simple group, the normal case actually seems significantly more difficult than we have seen before. Fortunately however, for many simple groups, the following result of Rose ensures that this sequence splits, see [17, Theorem 2.7].

Theorem 5.6. *If the center of G is trivial and the automorphism group of G splits over its inner automorphism group, then all extensions over G split.*

In particular, when the conditions of Theorem 5.6 are satisfied, such as with most alternating groups, then $A \cong K \ltimes G$, and finding all such groups of this form is significantly more tractable than the general case.

The non-normal case for simple groups

For the case $A \neq N$, the problem is purely computational with only finitely many solutions, so in principle, the groups and signatures can be calculated using GAP or MAGMA. In practice of course, complete classification is not likely since the strong branching cut-off for an arbitrary simple group is going to be quite large, and finite group databases do not typically include groups of high enough order. However, through additional ad hoc argumentation, and by restricting to intermediate extensions as outlined in Section 3, restrictions can be imposed on the possible groups and signatures which allow for steps to be made towards a more comprehensive classification. The following is an example of the types of computational results we can obtain to restrict our search.

Proposition 5.7. *If G is simple, $A \neq N$, and d is the index of N in A, then the number of periods r of the signature of A is bounded by:*

$$4\left(\frac{|G|-2}{d}+1\right) \geq r \geq 3.$$

Proof. Since $A \neq N$, we must have $\sigma \leq (|G|-1)^2$. Suppose that $(m_1, \ldots, m_r)$ is the signature of A. By the Riemann-Hurwitz formula,

$$\sigma - 1 = |A|\left(-1 + \frac{1}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right).$$

Using the bound on σ then gives us

$$\frac{|G|(|G|-2)}{|A|} \geq -1 + \frac{1}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right).$$

Rewrite this as

$$2\left(\frac{|G|-2}{d}+1\right) \geq \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right). \tag{56}$$

Since $m_i \geq 2$ for each i,

$$\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) \geq \frac{r}{2}$$

and thus

$$4\left(\frac{|G|-2}{d}+1\right) \geq r \geq 3.$$

□

Remark 5.3. We note that Proposition 5.7 actually holds provided the action of the group is weakly malnormal.

5.2.1. *Determining Automorphism Groups when $G = A_5$.* We finish by illustrating how such a classification might proceed by providing partial results for the first non-trivial case of this: when $G = A_5$. As is standard, we break the classification into two cases depending upon whether or not A_5 is normal in A.

The normal case when $G = A_5$

First we consider the case where $A = N$. Now we know any such group satisfies the short exact sequence

$$G \hookrightarrow A \twoheadrightarrow K$$

where K is one of the groups from Table 1. Moreover, since A_5 satisfies the hypotheses of Theorem 5.6, this sequence splits. In particular, A is a semidirect product which, for convenience, we consider as an outer semi-direct product $A_5 \rtimes_\psi K$ where $\psi\colon K \to \mathrm{Aut}(A_5) = S_5$ is the corresponding homomorphism defined by conjugation of elements of K on A_5. We can use these facts to determine all possible A's.

Proposition 5.8. *Suppose that $G = A_5$ and $A = N$. Then the possibilities for A are as follows:*

(1) *For all K, the direct product $A = A_5 \times K$.*
(2) *For $K = S_4$, $K = C_k$ for k even or $K = D_k$ for any k, we have an additional non-trivial semi-direct product $A_5 \rtimes_\psi K$ where $\psi\colon K \to Aut(A_5) = S_5$ is defined to have image $\langle(1,2)\rangle$ with kernel the unique index 2 subgroup of K.*
(3) *Further, for $K = D_k$, k even and $k > 2$, we have a second non-trivial semi-direct product $A_5 \rtimes K$ where $\psi\colon K \to Aut(A_5) = S_5$ is defined to have image $\langle(1,2)\rangle$ with kernel an index 2 dihedral group.*

Proof. Since $A = A_5 \rtimes_\psi K$, we just need to describe all such semi-direct products for each K. For $K = A_4$, S_4 and A_5, this can be done computationally using GAP and we attain the stated results. We need to explore in more detail the two infinite families $K = C_k$ and $K = D_k$

For $A = A_5 \rtimes_\psi K$, let K_1 be the kernel of ψ. Then the group $A_5 \rtimes_{\bar\psi} (K/K_1)$ where $\bar\psi$ is the induced homomorphism is either a non-trivial semi-direct product where $\bar\psi$ has trivial kernel, or it is isomorphic to A_5 and consequently A is a direct product $A_5 \times K$. Thus we consider $A_5 \rtimes_{\bar\psi} K/K_1$ for each possible K and K_1.

First suppose that $K = C_k$. Then K/K_1 is cyclic, and since it is a subgroup of $\mathrm{Aut}(A_5) = S_5$ it is either order 2, 3, 4 or 5. However, for each of these possibilities, simple computation using GAP gives a non-trivial semi-direct product $A_5 \rtimes_{\bar\psi} K/K_1$ only when K/K_1 is cyclic of order 2, and in this case $\bar\psi$ can be defined as in the statement of the theorem. The result follows.

Next suppose that $K = D_k$. Then K/K_1 is either cyclic of order 2 or dihedral. Therefore, since it is a subgroup of $\mathrm{Aut}(A_5) = S_5$ it is either cyclic of order 2 or dihedral of order 4, 6, 8, or 10. Again, for each of these possibilities, simple computation using GAP gives a non-trivial semi-direct product $A_5 \rtimes_{\bar\psi} K/K_1$ only when K/K_1 is cyclic of order 2 with the image of $\bar\psi$ as defined in the statement of the theorem. When k is odd, there is precisely one possible non-trivial kernel being the index 2 cyclic subgroup. When k is even and $k > 2$, there are three possible non-trivial kernels: the index two cyclic subgroup and the two index 2 dihedral subgroups, though the latter two yield isomorphic semi-direct products. The result follows.

□

The possible signatures with which the different normal extensions can act can be determined using Proposition 3.3. For a given such signature, determining whether or not an action exists depends on whether or not we can construct a generating vector, and this can in principle be done exhaustively. We note however that for a given A, there are many simple arguments which will eliminate signatures without having to consider generating vectors. Rather than presenting all possible signatures for all possible A, we illustrate with an example of how the general process follows, noting that for other A, the same general process holds. First, we fix the following notation.

Notation 5.9. *In a signature, we use the expression $m_i^{(k_i)}$ to denote k_i copies of the periods m_i.*

Example 5.9. Suppose that $K = C_2$, the cyclic group of order 2. Then there are two possibilities for A: the direct product $A_5 \times C_2$ and the semidirect product $A_5 \rtimes C_2$ which is isomorphic to the symmetric group S_5. Since the signature of K is $(2, 2)$, the possible signatures for each A are of the form

$$(2a_1, 2a_2, 2^{(a)}, 3^{(b)}, 5^{(c)}) \tag{57}$$

where $a_1, a_2 \in \{1, 2, 3, 5\}$. We now proceed by cases.

First suppose that $A = C_2 \times A_5$. Then A contains elements of orders $2, 3, 5, 6$ and 10, but not 4, so in particular, we can only have $a_1, a_2 \in \{1, 3, 5\}$. Of all remaining possible signatures of the form given in (57), the only ones for which a generating vector for A with n-gonal subgroup A_5 cannot be created are $(2 \cdot 3, 2 \cdot 3, 3)$ and $(2 \cdot 3, 2 \cdot 3, 2)$. Hence $A = C_2 \times A_5$ acts on an n-gonal surface S with n-gonal subgroup A_5 with all signatures of the form $(2a_1, 2a_2, 2^{(a)}, 3^{(b)}, 5^{(c)})$ for $a_1, a_2 \in \{1, 3, 5\}$ except $(2 \cdot 3, 2 \cdot 3, 3)$ and $(2 \cdot 3, 2 \cdot 3, 2)$.

Next suppose that $A = S_5$. Then A contains elements of order $2, 3, 4, 5$ and 6 but not 10 so in particular, we can only have $a_1, a_2 \in \{1, 2, 3\}$. Of all remaining possible signatures of the form given in (57), the only ones for which a generating vector for A with n-gonal subgroup A_5 cannot be created are $(2, 2, 3, 3)$ and $(2, 2, 2, 3)$. Hence $A = S_5$ acts on an n-gonal surface S with n-gonal subgroup A_5 with all signatures of the form $(2a_1, 2a_2, 2^{(a)}, 3^{(b)}, 5^{(c)})$ for $a_1, a_2 \in \{1, 2, 3\}$ except $(2, 2, 3, 3)$ and $(2, 2, 2, 3)$.

5.3. **The non-normal case when $G = A_5$.** Now suppose that $A \neq N$. Based on the computational evidence so far (see Example 3.5), there appear to be far more non-normal cases than we saw in the cyclic p-gonal case and at least currently there seems no obvious way to nicely categorize these non-normal extensions as we did in the cyclic p-gonal case. Therefore, rather than providing complete results, which currently seems computationally intractable, we shall provide some first steps to the general problem, and then illustrate with a few specific examples.

General facts for the non-normal case

By the strong branching condition, we know if $A \neq N$, then the genus of S must satisfy $\sigma < (|A_5| - 1)^2 = 3481$. Application of the Hurwitz bound then implies that the order of A and the index d of N in A satisfy

$$|A| \leq 84(\sigma - 1) = 292,320 \text{ and } d \leq 4872.$$

We know any signature for A_5 is of the form $(2^{(a)}, 3^{(b)}, 5^{(c)})$ for integers a, b and c so we can use the strong branching cut-off and the Riemann-Hurwitz formula to determine bounds on a, b and c. Specifically we have:

$$(60-1)^2 - 1 \geq \sigma - 1 = -60 + \frac{60}{2}\left(a\left(1 - \frac{1}{2}\right) + b\left(1 - \frac{1}{2}\right) + c\left(1 - \frac{1}{2}\right)\right)$$

which simplifies to

$$3540 \geq 15a + 20b + 24c.$$

It follows that

$$a \leq 230, \quad b \leq 172, \quad c \leq 147.$$

We note that not all choices of a, b and c are valid signatures and some may give a genus beyond the strong branching cut-off.

Next we observe by Proposition 5.7 that the number of periods of A satisfies

$$r \leq 4\left(\frac{58}{d}+1\right).$$

In particular, as d increases, the number of possible periods for A decreases, and in particular, when $d > 232$, then A has just three or four periods.

These conditions significantly reduce the number of possible non-normal extensions and the possible signatures, so at this point, to proceed with determining non-normal n-gonal extensions of A_5, we would follow the steps outlined in Section 3.

Determining alternating extensions of A_5

To illustrate our work, we consider n-gonal extensions of A_5 to all the alternating groups within the strong branching cut-off. Since $|A| \leq 292,320$, this gives A_6, A_7 and A_8 as possibilities. Also, we know that $A_5 < A_6 < A_7 < A_8$ with each containment being maximal, so the intermediate extensions are each primitive.

Consider first A_6 actions. The signature of an A_6 action will be of the form $(0; 2^{a_1}, 3^{b_1}, 4^{c_1}, 5^{d_1})$. As with A_5, we can use the strong branching cut-off and Riemann-Hurwitz formula to determine bounds on a_1, b_1, c_1 and d_1, see equation (5.3). Specifically, we get:

$$3480 \geq 360\left(-1+\frac{1}{2}\left(\frac{a_1}{2}+\frac{2b_1}{3}+\frac{3c_1}{4}+\frac{4d_1}{5}\right)\right)$$

which simplifies to

$$3840 \geq 90a_1 + 120b_1 + 135c_1 + 144d_1.$$

There are 38 164 solutions to this inequality, but with A_6 being relatively small in order, for each of these signatures, we can construct all possible generating vectors, and then apply Theorem 2.3 to each of the conjugacy classes of subgroups isomorphic to A_5 to check which ones are n-gonal. Using this process, we obtain 22 distinct solutions corresponding to actual signatures for n-gonal A_6 actions with a n-gonal A_5 subgroup. This is given in Table 4.

Sig(A_6)	Sig(A_5)	σ	Sig(A_6)	Sig(A_5)	σ
$(2,4,5)$	$(2^{(3)},5)$	10	$(2^{(5)})$	$(2^{(10)})$	91
$(3^{(2)},4)$	$(2,3^{(3)})$	16	$(2,3^{(3)})$	$(2^{(2)},3^{(6)})$	91
$(2,5^{(2)})$	$(2^{(2)},5^{(2)})$	19	$(2,3^{(2)},4)$	$(2^{(3)},3^{(6)})$	106
$(3^{(2)},5)$	$(3^{(3)},5)$	25	$(2,3^{(2)},5)$	$(2^{(2)},3^{(6)},5)$	115
$(3,4,5)$	$(2,3^{(3)},5)$	40	$(2^{(4)},3)$	$(2^{(8)},3^{(3)})$	121
$(2^{(3)},4)$	$(2^{(7)})$	46	$(3^{(3)},4)$	$(2,3^{(9)})$	136
$(3^{(2)},5)$	$(3^{(3)},5^{(2)})$	49	$(3^{(3)},5)$	$(3^{(9)},5)$	145
$(2^{(3)},5)$	$(2^{(6)},5)$	55	$(2^{(3)},3^{(2)})$	$(2^{(6)},3^{(6)})$	151
$(2^{(2)},3^{(2)})$	$(2^{(4)},3^{(3)})$	61	$(2^{(2)},3^{(3)})$	$(2^{(4)},3^{(9)})$	181
$(2^{(2)},3,4)$	$(2^{(5)},3^{(3)})$	76	$(2,3^{(4)})$	$(2^{(2)},3^{(12)})$	211
$(2^{(2)},3,5)$	$(2^{(4)},3^{(3)},5)$	85	$(3^{(5)})$	$(3^{(15)})$	241

Table 4. Signatures of n-gonal A_6 actions on surfaces of genus σ with corresponding signatures of n-gonal A_5 subgroups.

Remark 5.4. We note that in Table 4, we have only listed group signatures pairs and have not specified the number of distinct actions. In many cases, there are multiple actions up to the different types of equivalency, such as topological equivalence or simultaneous conjugation.

Next we consider n-gonal A_7 actions. Similar computations yield just 1021 possible signatures for A_7 that satisfy the strong branching cut-off. Once again, for each of these signatures, we can construct all possible generating vectors, and apply Theorem 2.3 to each of the conjugacy classes of subgroups isomorphic to A_5 to check which ones are n-gonal, and in this case, we obtain no possible solutions. In particular, there is no surface on which A_7 acts as an n-gonal group on which A_5 also acts as an n-gonal group.

Finally, since there are no solutions for A_7, and every subgroup of A_8 isomorphic to A_5 is contained in an intermediate subgroup isomorphic to A_7, there cannot be any solutions for A_8 either. Hence, the only alternating n-gonal extensions of an n-gonal group A_5 are given in Table 4.

Remark 5.5. One of the main results which allowed for complete results in determining non-normal extensions of cyclic p-gonal surfaces was the fact that when G is not the full automorphism group then G is not self-normalizing, see Corollary 3.3 of [22]. Since A_5 is maximal and non-normal in A_6, we see that this result does not extend to other groups.

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Department of Mathematics, Rose-Hulman Institute of Technology, 5500 Wabash Avenue Terre Haute, IN 47803
E-mail address: `brought@rose-hulman.edu`

Department of Mathematics, Oregon State University, Kidder Hall 368, Corvallis, OR 97331-4605
E-mail address: `camachoc@math.oregonstate.edu`

Department of Mathematics, Grinnell College, 1115 8th Avenue, Grinnell, IA 50112
E-mail address: `paulhus@math.grinnell.edu`

Department of Mathematics, University of Michigan, 2074 East Hall 530 Church Street, Ann Arbor, MI 48109-1043
E-mail address: `rebecca.winarski@gmail.com`

Department of Mathematics, University of Portland, 5000 N. Willamette Blvd., Portland, OR 97203
E-mail address: `wootton@up.edu`