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CHARACTER DEGREES OF GROUPS ASSOCIATED WITH FINITE SPLIT BASIC ALGEBRAS WITH INVOLUTION

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Dedicated to the memory of Kay Magaard

Abstract. Let $\mathcal{A}$ be a finite-dimensional split basic algebra over a finite field $\Bbbk$ with odd characteristic, and assume that $\mathcal{A}$ is endowed with an involution $\sigma\colon \mathcal{A} \to \mathcal{A}$. We determine the degrees of the irreducible characters of the group $C_G(\sigma) = \{x \in G\colon \sigma(x^{-1}) = x\}$ where $G = \mathcal{A}^\times$ is the unit group of $\mathcal{A}$, and prove that every irreducible character of $C_G(\sigma)$ is induced by a linear character of some subgroup. As a particular case, our results hold for the Sylow p-subgroups of the finite classical groups of Lie type, and extend (in a uniform way) the results obtained by B. Szegedy in [11].

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Let p be an odd prime, let $\Bbbk$ be a finite field of characteristic p, and let $\mathcal{A}$ be a finite-dimensional associative $\Bbbk$-algebra (with identity). We recall that an *involution* on $\mathcal{A}$ is a map $\sigma\colon \mathcal{A} \to \mathcal{A}$ satisfying the following conditions:

(1) $\sigma(a+b) = \sigma(a) + \sigma(b)$ for all $a, b \in \mathcal{A}$;
(2) $\sigma(ab) = \sigma(b)\sigma(a)$ for all $a, b \in \mathcal{A}$;
(3) $\sigma^2(a) = a$ for all $a \in \mathcal{A}$.

We note that an involution σ is not required to be $\Bbbk$-linear; however, we will assume that the field $\Bbbk = \Bbbk \cdot 1$ is preserved by σ. Then, σ defines a field automorphism of $\Bbbk$ which is either the identity or has order 2; we say that σ is *of the first kind* if σ fixes $\Bbbk$, and *of the second kind* if its restriction $\sigma_\Bbbk$ to $\Bbbk$ has order 2. In any case, we let $\Bbbk^\sigma = \{\alpha \in \Bbbk\colon \sigma(\alpha) = \alpha\}$ denote the σ-fixed subfield of $\Bbbk$, and consider $\mathcal{A}$ as a finite dimensional associative $\Bbbk^\sigma$-algebra. We observe that σ is of the second kind if and only if the field extension $\Bbbk^\sigma \subseteq \Bbbk$ has degree 2, and $\sigma\colon \Bbbk \to \Bbbk$ is the *Frobenius map* defined by the mapping $\alpha \mapsto \alpha^q$ where $q = |\Bbbk^\sigma|$; hence, $\Bbbk^\sigma = \mathbb{F}_q$ and $\Bbbk = \mathbb{F}_{q^2}$. For simplicity of writing, we will the bar notation $\overline{\alpha} = \alpha^q$ for $\alpha \in \Bbbk$.

Let $G = \mathcal{A}^\times$ denote the unit group of the $\Bbbk$-algebra $\mathcal{A}$. Then, for any involution $\sigma\colon \mathcal{A} \to \mathcal{A}$, the cyclic group $\langle\sigma\rangle$ acts on G as a group of automorphisms by means of $x^\sigma = \sigma(x^{-1})$ for all $x \in G$ (x^σ should not be confused with $\sigma(x)$). For any

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σ-invariant subgroup $H \leq G$, we denote by $C_H(\sigma)$ the subgroup of H consisting of all σ-fixed elements; that is,

$$C_H(\sigma) = \{x \in H \colon x^\sigma = x\} = \{x \in H \colon \sigma(x^{-1}) = x\}.$$

The main purpose of this paper is to determine the degree of any irreducible (complex) character of the group $C_G(\sigma)$ in the case where $\mathcal{A}$ is an arbitrary basic $\Bbbk$-algebra endowed with an involution $\sigma \colon \mathcal{A} \to \mathcal{A}$. By definition, a $\Bbbk$-algebra $\mathcal{A}$ is said to be *basic* if the Jacobson radical $\operatorname{Rad}(\mathcal{A}) \leq \mathcal{A}$ equals the set consisting of all nilpotent elements of $\mathcal{A}$; equivalently, the semisimple $\Bbbk$-algebra $\mathcal{A}/\operatorname{Rad}(\mathcal{A})$ is isomorphic to a direct sum $\Bbbk_1 \oplus \cdots \oplus \Bbbk_n$ of field extensions $\Bbbk_1, \ldots, \Bbbk_n$ of $\Bbbk$ (in the paper [10], B. Szegedy refers to $\mathcal{A}$ as an *N-algebra* over $\Bbbk$; see, in particular, [10, Lemma 2.1]). We note that every subalgebra (containing the identity) of a basic $\Bbbk$-algebra is also a basic $\Bbbk$-algebra; moreover, if $\mathcal{I}$ is any (two-sided) ideal of $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is also a basic $\Bbbk$-algebra. In the case where $\Bbbk_i \cong \Bbbk$ for all $1 \leq i \leq n$, we refer to $\mathcal{A}$ as a *split basic* $\Bbbk$*-algebra* (or, in the terminology of [10], as a *DN-algebra*); we observe that subalgebras (containing the identity) and quotient algebras of a split basic $\Bbbk$-algebra are also slit basic $\Bbbk$-algebras (see, for example, [10, Lemmas 2.2 and 2.3]).

As a standard example, let $\mathcal{M}_n(\Bbbk)$ be the full matrix algebra over $\Bbbk$ consisting of all $n \times n$ matrices with entries in $\Bbbk$, so that $\mathcal{M}_n(\Bbbk)^\times = \mathrm{GL}_n(\Bbbk)$ is the general linear group consisting of all invertible matrices in $\mathcal{M}_n(\Bbbk)$. The $\Bbbk$-algebra $\mathcal{M}_n(\Bbbk)$ is canonically endowed with the *transpose involution* defined by the mapping $a \mapsto a^\mathsf{T}$ where a^T denotes the transpose of $a \in \mathcal{M}_n(\Bbbk)$. Let $q = |\Bbbk^\sigma|$, let $F \colon \mathcal{M}_n(\Bbbk) \to \mathcal{M}_n(\Bbbk)$ be the Frobenius morphism defined by $F(a_{ij}) = (\overline{a_{ij}}) = ({a_{ij}}^q)$ for all $(a_{ij}) \in \mathcal{M}_n(\Bbbk)$, and set $a^* = F(a)^\mathsf{T}$ for all $a \in \mathcal{M}_n(\Bbbk)$. Then, the mapping $a \mapsto a^*$ defines an involution on $\mathcal{M}_n(\Bbbk)$; notice that, if $\Bbbk^\sigma = \Bbbk$, then $a^* = a^\mathsf{T}$ for all $a \in \mathcal{M}_n(\Bbbk)$. If $\sigma \colon \mathcal{M}_n(\Bbbk) \to \mathcal{M}_n(\Bbbk)$ is an involution of the first kind, then there exists $u \in \mathrm{GL}_n(\Bbbk)$ with $u^\mathsf{T} = \pm u$ and such that $\sigma(a) = u^{-1}a^\mathsf{T}u$ for all $a \in \mathcal{M}_n(\Bbbk)$; moreover, the matrix u is uniquely determined up to a factor in $\Bbbk^\times$. On the other hand, if $\sigma \colon \mathcal{M}_n(\Bbbk) \to \mathcal{M}_n(\Bbbk)$ is an involution of the second kind, then there exists $u \in \mathrm{GL}_n(\Bbbk)$ with $u^* = u$ and such that $\sigma(a) = u^{-1}a^*u$ for all $a \in \mathcal{M}_n(\Bbbk)$; moreover, the matrix u is uniquely determined up to a factor in $(\Bbbk^\sigma)^\times$. (The proofs can be found in the book [8] by M.-A. Knus *et al.* where the complete classification of involutions is also given for arbitrary central $\Bbbk$-algebras.) For simplicity, for $u \in \mathrm{GL}_n(\Bbbk)$ as above, we will denote by σ_u the involution on $\mathcal{M}_n(\Bbbk)$ given by the mapping $a \mapsto u^{-1}a^*u$; as usual, we say that σ_u is *symplectic* if σ_u is of the first kind and $u^\mathsf{T} = -u$, *orthogonal* if σ_u is of the first kind and $u^\mathsf{T} = u$, and *unitary* if σ_u is of the second kind and $u^* = u$.

For an arbitrary involution $\sigma \colon \mathcal{M}_n(\Bbbk) \to \mathcal{M}_n(\Bbbk)$ the group $C_{\mathrm{GL}_n(\Bbbk)}(\sigma)$ is isomorphic to one of the well-known *finite classical groups of Lie type* (defined over $\Bbbk$): the *symplectic group* $\mathrm{Sp}_{2m}(q)$ if σ is symplectic (and $\Bbbk = \mathbb{F}_q$), the *orthogonal groups* $\mathrm{O}^+_{2m}(q)$, $\mathrm{O}_{2m+1}(q)$, or $\mathrm{O}^-_{2m+2}(q)$ if σ is orthogonal (and $\Bbbk = \mathbb{F}_q$), and the *unitary group* $\mathrm{U}_n(q^2)$ if σ is unitary (and $\Bbbk = \mathbb{F}_{q^2}$). (For the details on the definition of the classical groups, we refer to Chapter I the book [2] by R. Carter.) In fact, up to isomorphism, these groups may be defined by the involution $\sigma = \sigma_u$ where $u \in \mathrm{GL}_n(\Bbbk)$ is defined as follows; here, J_m denotes the $m \times m$ matrix with 1's along the anti-diagonal and 0's elsewhere.

(1) For $\mathrm{Sp}_{2m}(q)$, we choose $\Bbbk = \mathbb{F}_q$ and $u = \left(\begin{smallmatrix} 0 & J_m \\ -J_m & 0 \end{smallmatrix}\right)$.

(2) For $O_{2m}^{+}(q)$ or $O_{2m+1}(q)$, we choose $\Bbbk = \mathbb{F}_q$ and $u = J_n$ where, either $n = 2m$, or $n = 2m + 1$.
(3) For $O_{2m+2}^{-}(q)$, we choose $\Bbbk = \mathbb{F}_q$ and $u = \left(\begin{smallmatrix} 0 & 0 & J_m \\ 0 & c & 0 \\ J_m & 0 & 0 \end{smallmatrix}\right)$ where $c = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -\varepsilon \end{smallmatrix}\right)$ for $\varepsilon \in \mathbb{F}_q^{\times} - (\mathbb{F}_q^{\times})^2$.
(4) For $U_n(q^2)$, we choose $\Bbbk = \mathbb{F}_{q^2}$ and $u = J_n$. (In this case, we have $\Bbbk^{\sigma} = \mathbb{F}_q$.)

Let $\mathcal{A} = \mathfrak{b}_n(\Bbbk)$ be the *Borel subalgebra* of $\mathcal{M}_n(\Bbbk)$ consisting of all upper-triangular matrices; hence, $G = \mathcal{A}^{\times}$ is the standard Borel subgroup $B_n(\Bbbk)$ of $\mathrm{GL}_n(\Bbbk)$. Then, $\mathcal{A}$ is a split basic $\Bbbk$-algebra; in fact, the Jacobson radical $\mathrm{Rad}(\mathcal{A})$ is the (upper) niltriangular subalgebra $\mathfrak{ut}_n(\Bbbk) \leq \mathfrak{b}_n(\Bbbk)$ consisting of all upper-triangular matrices with 0's on the main diagonal, and $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is isomorphic to a direct sum of n copies of $\Bbbk$; indeed, $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is isomorphic to the diagonal subalgebra $\mathfrak{d}_n(\Bbbk)$ consisting of all diagonal matrices in $\mathcal{M}_n(\Bbbk)$. Further, $\mathcal{A}$ is a σ-invariant subalgebra of $\mathcal{M}_n(\Bbbk)$, and the $C_G(\sigma)$ is a (standard) Borel subgroup of the corresponding finite classical group.

In the general situation, let $\mathcal{A}$ be a split basic $\Bbbk$-algebra with an involution $\sigma\colon \mathcal{A} \to \mathcal{A}$. For any (nilpotent) subalgebra $\mathcal{J}$ of $\mathrm{Rad}(\mathcal{A})$, the set $1+\mathcal{J}$ is a p-subgroup of the unit group $G = \mathcal{A}^{\times}$ to which we refer as an *algebra subgroup* of G (as defined in [6]). In the particular case where $\mathcal{J} = \mathrm{Rad}(\mathcal{A})$, it is clear that $P = 1 + \mathrm{Rad}(\mathcal{A})$ is a normal subgroup of G, and that it is the unique Sylow p-subgroup of G. Furthermore, G is the semidirect product $G = TP$ where $T \leq G$ is isomorphic to the unit group of $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$; hence, T is isomorphic to the direct product $\Bbbk_1^{\times} \times \cdots \times \Bbbk_n^{\times}$ where $\Bbbk_1, \ldots, \Bbbk_n$ are field extensions of $\Bbbk$ such that $\mathcal{A}/\mathrm{Rad}(\mathcal{A}) \cong \Bbbk_1 \oplus \cdots \oplus \Bbbk_n$. Since $\mathcal{A}$ is split, we have $\Bbbk_i \cong \Bbbk$ for all $1 \leq i \leq n$, and in fact there are nonzero orthogonal idempotents $e_1, \ldots, e_n \in \mathcal{A}$ with $1 = e_1 + \cdots + e_n$, and such that $\mathcal{A} = \mathcal{D} \oplus \mathrm{Rad}(\mathcal{A})$ for $\mathcal{D} = \Bbbk e_1 \oplus \cdots \oplus \Bbbk e_n$; this follows easily from the usual process of "lifting idempotents" (see, for example, [9, Chapter VII]; see also [5, Lemma 2.1]). Then, $T = \mathcal{D}^{\times}$ is the unit group of the subalgebra $\mathcal{D}$; we will refer to $\mathcal{D}$ as the *diagonal subalgebra* of $\mathcal{A}$, and to T as the *diagonal subgroup* of $G = \mathcal{A}^{\times}$. In particular, we have $|G| = |\Bbbk|^r (|\Bbbk| - 1)^n$ where $r = \dim \mathrm{Rad}(\mathcal{A})$.

On the other hand, let $x \in G$ be arbitrary, and denote by $C_G(x)$ the centraliser of x in G (with respect to conjugation). It is clear that $C_G(x)$ is the unit group of the subalgebra $C_{\mathcal{A}}(x) = \{a \in \mathcal{A} \colon ax = xa\}$ of $\mathcal{A}$. Since every subalgebra of a split basic $\Bbbk$-algebra is also a split basic $\Bbbk$-algebra (see [10, Lemma 2.2]), $C_{\mathcal{A}}(x)$ is a split basic $\Bbbk$-algebra, and thus $|C_G(x)| = |\Bbbk|^s (|\Bbbk| - 1)^m$ for some nonnegative integers s and m (with $s \leq r$ and $m \leq n$). Since $(|\Bbbk|, |\Bbbk| - 1) = 1$, we deduce the following result.

Theorem 1 (Szegedy; see [10, Lemma 2.4])**.** *Let $\mathcal{A}$ be a split basic $\Bbbk$-algebra, let $G = \mathcal{A}^{\times}$, and let $\mathcal{K}$ be a conjugacy class of G. Then, $|\mathcal{K}| = |\Bbbk|^k (|\Bbbk| - 1)^l$ for some nonnegative integers k and l.*

Next, we consider the involution $\sigma\colon \mathcal{A} \to \mathcal{A}$, and determine the order of the σ-fixed subgroup $C_G(\sigma)$. We start by proving the following elementary result.

Lemma 1. *Let $\mathcal{A}$ be a $\Bbbk$-algebra with an involution $\sigma\colon \mathcal{A} \to \mathcal{A}$, let $\mathcal{J}$ be a σ-invariant nilpotent subalgebra of $\mathcal{A}$, and let $Q = 1 + \mathcal{J}$. Then, $|C_Q(\sigma)|$ is a power of $|\Bbbk^{\sigma}|$.*

Proof. Let $\varphi\colon \mathcal{J} \to Q$ be the Cayley transform defined by $\varphi(a) = (1 - a)(1 + a)^{-1}$ for all $a \in \mathcal{J}$. Since p is odd, the map φ is bijective, and it is easy to check that

$C_Q(\sigma) = \varphi(C_{\mathcal{J}}(\sigma))$ where $C_{\mathcal{J}}(\sigma) = \{a \in \mathcal{J} \colon \sigma(a) = -a\}$. The result follows because $C_{\mathcal{J}}(\sigma)$ is a vector space over $\Bbbk^\sigma$. ☐

On the other hand, we have the following.

Theorem 2. *Let $\mathcal{A}$ be a split basic $\Bbbk$-algebra with an involution $\sigma \colon \mathcal{A} \to \mathcal{A}$, let $G = \mathcal{A}^\times$ be the unit group of $\mathcal{A}$, and let $P = 1 + \mathrm{Rad}(\mathcal{A})$. Let $\Bbbk^\sigma$ be the σ-fixed field of $\Bbbk$, and let $q = |\Bbbk^\sigma|$. Then, $C_G(\sigma)/C_P(\sigma) \cong H \times K$ where H is a direct product of copies of $\Bbbk^\times$, and K is a direct product of cyclic groups of order $(q-1)/2$ if σ is of the first kind, and $q-1$ if σ is of the second kind. In particular, there exist nonnegative integers k and r such that*

$$|C_G(\sigma) : C_P(\sigma)| = \begin{cases} 2^{-k}(q-1)^r, & \text{if } \sigma \text{ is of the first kind,} \\ (q+1)^k(q-1)^r, & \text{if } \sigma \text{ is of the second kind.} \end{cases}$$

Further, we have $C_G(\sigma)P/P = C_{G/P}(\sigma)$.

Proof. Let $e_1, \ldots, e_n \in \mathcal{A}$ be nonzero orthogonal idempotents, and consider the diagonal subalgebra $\mathcal{D} = \Bbbk e_1 \oplus \cdots \oplus \Bbbk e_n$ of $\mathcal{A}$; moreover, for simplicity, we set $\mathcal{J} = \mathrm{Rad}(\mathcal{A})$.

Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$. Since $\sigma(e_1), \cdots, \sigma(e_n)$ are nonzero orthogonal idempotents satisfying $1 = \sigma(e_1) + \cdots + \sigma(e_n)$, there exist a permutation $\pi \in S_n$ and an invertible element $x \in P = 1 + \mathcal{J}$ such that $\sigma(e_i) = x e_{\pi(i)} x^{-1}$ for all $1 \le i \le n$ (see, for example, [9, Theorem VII.13]). In particular, we see that $\sigma(e_i) \in e_{\pi(i)} + \mathcal{J}$, and thus $\sigma(\Bbbk e_i) = \Bbbk\sigma(e_i) \subseteq \Bbbk e_{\pi(i)} + \mathcal{J}$ for all $1 \le i \le n$. Moreover, since σ is an involution, we clearly have $\pi^2 = 1$.

The involution $\sigma \colon \mathcal{A} \to \mathcal{A}$ defines naturally an involution on the $\Bbbk$-algebra $\mathcal{A}/\mathcal{J}$; if we denote this involution also by σ, then $\sigma(a + \mathcal{J}) = \sigma(a) + \mathcal{J}$ for all $a \in \mathcal{A}$. Hence, σ defines an automorphism of the group $G/P \cong (A/\mathcal{J})^\times$ by means of $(xP)^\sigma = x^\sigma P$ for all $x \in G$. Since $\mathcal{A} = \mathcal{D} \oplus \mathcal{J}$, we have $\mathcal{A}/\mathcal{J} \cong \mathcal{D}$, and thus $G/P \cong T$ where $T = \mathcal{D}^\times$ is the diagonal subgroup of G. For every $t \in T$, we have $tP \in C_{G/P}(\sigma)$ if and only if $t^{-1}t^\sigma \in P$, and so $C_{G/P}(\sigma) = \{tP \colon t^{-1}t^\sigma \in P\}$. On the other hand, since $\mathcal{D} = \Bbbk e_1 \oplus \cdots \oplus \Bbbk e_n$, every element of $t \in T = \mathcal{D}^\times$ is uniquely expressed as a sum $t = \alpha_1 e_1 + \cdots + \alpha_n e_n$ where $\alpha_1, \ldots, \alpha_n \in \Bbbk^\times$. In particular, for every $1 \le i \le n$ and every $\alpha \in \Bbbk^\times$, the element

$$t_i(\alpha) = \alpha e_i + \sum_{1 \le j \ne i \le n} e_i$$

lies in T; indeed, every $t \in T$ factorises uniquely as a product $t = t_1(\alpha_1) \cdots t_n(\alpha_n)$ where $\alpha_1, \ldots, \alpha_n \in \Bbbk^\times$. For every $1 \le i \le n$, let $T_i = \{t_i(\alpha) \colon \alpha \in \Bbbk^\times\}$; notice that $T_1, \ldots, T_n$ are subgroups of T and that T is the (internal) direct product $T = T_1 \cdots T_n$. Similarly, if we define $\overline{T}_i = T_i P/P$ for all $1 \le i \le n$, then G/P is the direct product $G/P = \overline{T}_1 \cdots \overline{T}_n$; moreover, since $\sigma(\Bbbk e_i) \subseteq \Bbbk e_{\pi(i)} + \mathcal{J}$, we must have $(\overline{T}_i)^\sigma \subseteq \overline{T}_{\pi(i)}$, and hence $(\overline{T}_i)^\sigma = \overline{T}_{\pi(i)}$ for all $1 \le i \le n$.

Now, if $t \in T$ is arbitrary and $t = t_1 \cdots t_n$ where $t_i \in T_i$ for all $1 \le i \le n$, then $t^{-1}t^\sigma = (t_1^{-1}(t_{\pi(1)})^\sigma) \cdots (t_n^{-1}(t_{\pi(n)})^\sigma)$ where $t_i^{-1}(t_{\pi(i)})^\sigma \in T_i P$ for all $1 \le i \le n$, and so $t^{-1}t^\sigma \in P$ if and only if $t_i^{-1}(t_{\pi(i)})^\sigma \in P$ for all $1 \le i \le n$; in other words, we have $tP \in C_{G/P}(\sigma)$ if and only if $t_i^{-1}(t_{\pi(i)})^\sigma \in P$ for all $1 \le i \le n$. In particular, if we set $\bar{t}_i(\alpha) = t_i(\alpha)P$, then $\bar{t}_i(\alpha)\bar{t}_i(\alpha)^\sigma \in C_{G/P}(\sigma)$ for all $\alpha \in \Bbbk^\times$ and all $1 \le i \le n$. In fact, it is straightforward to check that, for all $1 \le i \le n$, the mapping $\alpha \mapsto \bar{t}_i(\alpha)\bar{t}_i(\alpha)^\sigma$ defines a group homomorphism $\gamma_i\colon \Bbbk^\times \to C_{G/P}(\sigma)$, and that $C_{G/P}(\sigma) = \prod_{i\in I} \operatorname{Im}(\gamma_i)$ where I is a complete set of representatives of the π-orbits on $\{1,2,\dots,n\}$. In particular, we conclude that

$$|C_{G/P}(\sigma)| = \prod_{i\in I} |\operatorname{Im}(\gamma_i)|.$$

It is clear that γ_i is injective whenever $i \in I$ is such that $\pi(i) \neq i$. On the other hand, let $i \in I$ be such that $\pi(i) = i$. In this case, $(\Bbbk e_i + \mathcal{J})/\mathcal{J} = \Bbbk \bar{e}_i$ where $\bar{e}_i = e_i + \mathcal{J}$, and we have $\sigma(\alpha\bar{e}_i) = \alpha^q e_i + \mathcal{J}$ for all $\alpha \in \Bbbk$. In particular, for any $\alpha \in \Bbbk^\times$, we deduce that $\alpha \in \ker(\gamma_i)$ if and only if $\alpha = \alpha^q$, and so

$$|\operatorname{Im}(\gamma_i)| = \begin{cases} q-1, & \text{if } \Bbbk^\sigma = \Bbbk,\\ (q-1)/2, & \text{if } \Bbbk^\sigma \neq \Bbbk.\end{cases}$$

Furthermore, we conclude that $C_{G/P}(\sigma)$ is isomorphic to a direct product $H \times K$ where H is a direct product of copies of $\Bbbk^\times$, and K is a direct product of cyclic groups of order $(q-1)/2$ if σ is of the first kind, or $q-1$ if σ is of the second kind. In particular, there exist nonnegative integers k and r such that

$$|C_{G/P}(\sigma)| = \begin{cases} 2^{-k}(q-1)^r, & \text{if } \sigma \text{ is of the first kind},\\ (q+1)^k(q-1)^r, & \text{if } \sigma \text{ is of the second kind}.\end{cases}$$

If we assume further that the diagonal subalgebra $\mathcal{D} \leq \mathcal{A}$ is σ-invariant, we clearly have a semidirect product $C_G(\sigma) = C_T(\sigma)C_P(\sigma)$ where $T = \mathcal{D}^\times$, and thus $C_{G/P}(\sigma) \cong C_T(\sigma) \cong C_G(\sigma)/C_P(\sigma)$. Therefore, in this situation, we conclude that there exist nonnegative integers k and r such that

$$|C_G(\sigma) : C_P(\sigma)| = \begin{cases} 2^{-k}(q-1)^r, & \text{if } \sigma \text{ is of the first kind},\\ (q+1)^k(q-1)^r, & \text{if } \sigma \text{ is of the second kind}.\end{cases}$$

In the general situation, let $\widetilde{G}$ be the semidirect product $\widetilde{G} = G \ltimes \langle\sigma\rangle$ of G by the cyclic group $\langle\sigma\rangle$. Since $\widetilde{G}$ is solvable and $\sigma \in \widetilde{G}$ has order 2, Hall's Theorem (see [3, Theorem 6.41]) asserts that there exists a Hall p'-subgroup $\widetilde{S} \leq \widetilde{G}$ with $\sigma \in \widetilde{S}$. Then, $S = \widetilde{S} \cap G$ is a Hall p'-subgroup of G, and we have $G = PS$ (by order considerations); moreover, since $\sigma \in \widetilde{S}$, the subgroup S is clearly σ-invariant. It follows that $C_G(\sigma)$ is the semidirect product $C_G(\sigma) = C_P(\sigma)C_S(\sigma)$, and hence $C_G(\sigma)P/P \cong C_S(\sigma) \cong C_{G/P}(\sigma)$. □

We are now able to determine the size of any conjugacy class of $C_G(\sigma)$.

Theorem 3. *Let $\mathcal{A}$ be a split basic $\Bbbk$-algebra with an involution $\sigma\colon \mathcal{A} \to \mathcal{A}$, let $G = \mathcal{A}^\times$, and let $\mathcal{K}$ be a conjugacy class of $C_G(\sigma)$. Then, there exist nonnegative integers k, r and s such that*

$$|\mathcal{K}| = \begin{cases} 2^{-k}(q-1)^r q^s, & \textit{if } \sigma \textit{ is of the first kind},\\ (q+1)^k(q-1)^r q^s, & \textit{if } \sigma \textit{ is of the second kind},\end{cases}$$

where $q = |\Bbbk^\sigma|$.

Proof. Let $x \in \mathcal{K}$ be arbitrary, and recall that $C_G(x)$ is the unit group $H = \mathcal{B}^\times$ of the subalgebra $\mathcal{B} = C_{\mathcal{A}}(x)$ of $\mathcal{A}$. Since $x \in C_G(\sigma)$, it is clear that $\mathcal{B}$ is σ-invariant. Since $C_H(\sigma) = H \cap C_G(\sigma)$, we have $|\mathcal{K}| = |C_G(\sigma) : C_H(\sigma)|$, and thus

$$|\mathcal{K}| = |C_G(\sigma) : C_P(\sigma)|\, |C_H(\sigma) : C_Q(\sigma)|^{-1} |C_P(\sigma) : C_Q(\sigma)|$$

where $Q = P \cap H = 1 + \mathrm{Rad}(\mathcal{B})$. The result follows by Lemma 1 and by the previous theorem. $\square$

Next, we consider the irreducible characters of $C_G(\sigma)$. Our goal is to prove the following main result. (We observe that, in the case where σ is an involution of the first kind, this result is essentially [11, Theorem 6].)

Theorem 4. *Let $\mathcal{A}$ be a split basic $\Bbbk$-algebra with an involution $\sigma\colon \mathcal{A} \to \mathcal{A}$, let $G = \mathcal{A}^\times$ be the unit group of $\mathcal{A}$, and let χ be an arbitrary irreducible character of $C_G(\sigma)$. Then, there exist nonnegative integers k, r and s such that*

$$\chi(1) = \begin{cases} 2^{-k}(q-1)^r q^s, & \text{if } \sigma \text{ is of the first kind,} \\ (q+1)^k (q-1)^r q^s, & \text{if } \sigma \text{ is of the second kind,} \end{cases}$$

where $q = |\Bbbk^\sigma|$.

The following reduction result will be crucial for the proof of this theorem. As usual, given an arbitrary function $\chi\colon G \to \mathbb{C}$ of a group G and an arbitrary element $g \in G$, we define the function $\chi^g\colon G \to \mathbb{C}$ by the rule $\chi^g(x) = \chi(gxg^{-1})$ for all $x \in G$; similarly, given an arbitrary subset $\mathcal{X}$ of G and an arbitrary element $g \in G$, we define $\mathcal{X}^g = \{x^g \colon x \in \mathcal{X}\}$ where $x^g = gxg^{-1}$ for all $x \in G$.

Theorem 5. *Let $\mathcal{A}$ be a split basic $\Bbbk$-algebra with an involution $\sigma\colon \mathcal{A} \to \mathcal{A}$, let $G = \mathcal{A}^\times$ be the unit group of $\mathcal{A}$, and let $P = 1 + \mathrm{Rad}(\mathcal{A})$. Let χ be a σ-invariant irreducible character of P, and let $I_G(\chi) = \{g \in G \colon \chi^g = \chi\}$ be the inertia group of χ. Then, $I_G(\chi) = \mathcal{B}^\times$ for some σ-invariant subalgebra $\mathcal{B} \leq \mathcal{A}$.*

Proof. Let $\widetilde{G}$ be the semidirect product $\widetilde{G} = G \ltimes \langle\sigma\rangle$ of G by the cyclic group $\langle\sigma\rangle$. Since $P = 1 + \mathrm{Rad}(\mathcal{A})$ is σ-invariant, P is a normal subgroup of $\widetilde{G}$. As in the proof of Theorem 2, we may choose a Hall p'-subgroup $S \leq \widetilde{G}$ with $\sigma \in S$ and such that $\widetilde{G}$ is the semidirect product $\widetilde{G} = PS$.

The group S acts naturally on the set $\mathrm{Irr}(P)$ of irreducible characters of P and on the set $\mathrm{Cl}(P)$ of conjugacy classes of P. By [7, Theorem 13.24], these actions are permutation isomorphic. Let $\beta\colon \mathrm{Irr}(P) \to \mathrm{Cl}(P)$ be a S-equivariant bijection, and let $\mathcal{K} = \beta(\chi)$. Then, $C_S(\chi) = \{s \in S \colon \mathcal{K}^s = \mathcal{K}\}$. Since $C_S(\chi)$ is a p'-group, Glauberman's Lemma (see [7, Lemma 13.8]) implies that there exists $x \in \mathcal{K}$ such that $x^s = x$ for all $s \in C_S(\chi)$; in particular, since χ is σ-invariant, we have $\sigma \in C_S(\chi)$, and thus $x^\sigma = x$.

We now claim that $I_G(\chi) = PC_G(x)$. In fact, let $g \in G$ be arbitrary. Since $\widetilde{G} = PS$, there are uniquely determined elements $h \in P$ and $s \in S \cap G$ such that $g = hs$; thus, we have $\mathcal{K}^g = \mathcal{K}^s$ and $\chi^g = \chi^s$. On the one hand, suppose that $g \in C_G(x)$. Then, $\mathcal{K}^s = \mathcal{K}^g = \mathcal{K}$, and so $s \in C_{S\cap G}(\chi) \leq I_G(\chi)$. On the other hand, suppose that $g \in I_G(\chi)$. Then, $\chi^s = \chi^g = \chi$, and so $s \in C_S(\chi)$. By the choice of x, we conclude that $s \in C_G(x)$, and thus $g = hs \in PC_G(x)$. The claim follows.

To complete the proof it is enough to take $\mathcal{B} = C_{\mathcal{A}}(x) + \mathrm{Rad}(\mathcal{A})$ where $C_{\mathcal{A}}(x) = \{a \in \mathcal{A} \colon xa = ax\}$; it is clear that $\mathcal{B}$ is a σ-invariant subalgebra of $\mathcal{A}$, and that $\mathcal{B}^\times = PC_G(x) = I_G(x)$. $\square$

We now proceed with the proof of Theorem 4.

Proof of Theorem 4. We start by recalling the Glauberman correspondence between σ-invariant irreducible characters of $P = 1 + \mathrm{Rad}(\mathcal{A})$ and irreducible characters of $C_P(\sigma)$; our main reference is [7, Chapter 13]. As usual, we denote by $\mathrm{Irr}(P)$ the set consisting of all irreducible characters of P (and extend this notation to any finite group), and by $\mathrm{Irr}_\sigma(P)$ the subset of $\mathrm{Irr}(P)$ consisting of all σ-invariant irreducible characters. Since p is odd, the Glauberman correspondence asserts that there exists a uniquely defined bijective map $\pi_P\colon \mathrm{Irr}_\sigma(P) \to \mathrm{Irr}(C_P(\sigma))$ such that, for any $\widehat{\varphi} \in \mathrm{Irr}_\sigma(P)$, the image $\varphi = \pi_P(\widehat{\varphi})$ is the unique irreducible constituent of the restriction $\widehat{\varphi}_{C_P(\sigma)}$ which occurs with odd multiplicity (see [7, Theorem 13.1]).

Now, let χ be an arbitrary irreducible character of $C_G(\sigma)$, let $\varphi \in \mathrm{Irr}(C_P(\sigma))$ be an irreducible constituent of $\chi_{C_P(\sigma)}$, and let $\widehat{\varphi} \in \mathrm{Irr}_\sigma(P)$ be such that $\pi_P(\widehat{\varphi}) = \varphi$. We consider the inertia group $I_G(\widehat{\varphi})$ of $\widehat{\varphi}$, and observe that

$$I_{C_G(\sigma)}(\varphi) = I_G(\widehat{\varphi}) \cap C_G(\sigma).$$

In fact, let $g \in C_G(\sigma)$ be arbitrary. Then, it is clear that $\widehat{\varphi}^{\,g} \in \mathrm{Irr}_\sigma(P)$; moreover, we have $\pi_P(\widehat{\varphi}^{\,g}) = \varphi^g$ (by [7, Theorem 13.1] because $\langle \varphi^g, (\widehat{\varphi}^{\,g})_{C_P(\sigma)} \rangle = \langle \varphi, \widehat{\varphi}_{C_P(\sigma)} \rangle$). Since π_P is bijective, we conclude that $\widehat{\varphi}^{\,g} = \widehat{\varphi}$ if and only if $\varphi^g = \varphi$. On the other hand, by Theorem 5, $I_G(\widehat{\varphi})$ is the unit group $H = \mathcal{B}^\times$ of some subalgebra $\mathcal{B} \leq \mathcal{A}$; we note that $\mathrm{Rad}(\mathcal{B}) = \mathrm{Rad}(\mathcal{A})$. By Theorem 2, we conclude that there are nonnegative integers k and r such that

$$|C_G(\sigma) : I_{C_G(\sigma)}(\varphi)| = \begin{cases} 2^{-k}(q-1)^r, & \text{if } \sigma \text{ is of the first kind,} \\ (q+1)^k(q-1)^r, & \text{if } \sigma \text{ is of the second kind;} \end{cases}$$

in fact, $I_{C_G(\sigma)}(\varphi) = C_G(\sigma) \cap I_G(\widehat{\varphi}) = C_G(\sigma) \cap H = C_H(\sigma)$. Since χ is an irreducible constituent of $\varphi^{C_G(\sigma)}$, Clifford correspondence (see [7, Theorem 6.11]) implies that $\chi = \psi^{C_G(\sigma)}$ for some irreducible character ψ of $I_{C_G(\sigma)}(\chi) = C_H(\sigma)$, and hence

$$\chi(1) = \begin{cases} 2^{-k}(q-1)^r\psi(1), & \text{if } \sigma \text{ is of the first kind,} \\ (q+1)^k(q-1)^r\psi(1), & \text{if } \sigma \text{ is of the second kind.} \end{cases}$$

Since $p \nmid |C_H(\sigma) : C_P(\sigma)|$, [7, Corollary 6.28] implies that φ is extendible to $C_H(\sigma)$; in other words, there exists $\psi' \in \mathrm{Irr}(C_H(\sigma))$ such that $\psi'_{C_P(\sigma)} = \varphi$. Since $C_H(\sigma)/C_P(\sigma)$ is abelian, we have

$$\varphi^{C_H(\sigma)} = \sum_{\omega \in \mathrm{Irr}(C_H(\sigma)/C_P(\sigma))} \omega\psi'$$

(by Gallagher's Theorem; see [7, Corollary 6.17]), and so $\psi = \omega\psi'$ for some $\omega \in \mathrm{Irr}(C_H(\sigma))$ with $C_P(\sigma) \subseteq \ker(\omega)$. It follows that $\psi_{C_P(\sigma)} = \varphi$, and hence ψ is an also extension of φ. Therefore,

$$\chi(1) = \begin{cases} 2^{-k}(q-1)^r\varphi(1), & \text{if } \sigma \text{ is of the first kind,} \\ (q+1)^k(q-1)^r\varphi(1), & \text{if } \sigma \text{ is of the second kind.} \end{cases}$$

The proof of Theorem 4 is complete because $\varphi(1)$ is a power of q (by [1, Theorem 1.3]; see also [11, Theorem 1]). □

Finally, we prove that $C_G(\sigma)$ is in fact an *M-group*; that is, every irreducible character $\chi \in \mathrm{Irr}(C_G(\sigma))$ is induced by a linear character of some subgroup of $C_G(\sigma)$. More precisely, we shall prove the following result. (For a particular situation, see [11, Theorem 4].)

Theorem 6. *Let $\mathcal{A}$ be a split basic $\mathbb{k}$-algebra with an involution $\sigma\colon \mathcal{A} \to \mathcal{A}$, let $G = \mathcal{A}^\times$ be the unit group of $\mathcal{A}$, and let χ be an irreducible character of $C_G(\sigma)$. Then, there exist a σ-invariant subgroup $H \leq G$ and a linear character ϑ of $C_H(\sigma)$ such that $\chi = \vartheta^{C_G(\sigma)}$.*

Proof. Let $P = 1 + \mathcal{J}$ where $\mathcal{J} = \mathrm{Rad}(\mathcal{A})$, let $\varphi \in \mathrm{Irr}(C_P(\sigma))$ be an irreducible constituent of the restriction $\chi_{C_P(\sigma)}$, and let $\widehat{\varphi} \in \mathrm{Irr}_\sigma(P)$ be the Glauberman correspondent of φ. By Theorem 5 and by the proof of Theorem 4, we may assume that $\widehat{\varphi}$ is G-invariant; hence, φ is also $C_G(\sigma)$-invariant, and we have $\chi_{C_G(\sigma)} = \varphi$ (see the proof of Theorem 4). As in the proof of Theorem 2, let $\widetilde{G}$ be the semidirect product $\widetilde{G} = G \ltimes \langle\sigma\rangle$ of G by the cyclic group $\langle\sigma\rangle$, and let $\widetilde{S}$ be a Hall p'-subgroup of $\widetilde{G}$ with $\sigma \in \widetilde{S}$. Then, $S = G \cap \widetilde{S}$ is a σ-invariant Hall p'-subgroup of G, and we have a semidirect product $G = PS$; on the other hand, $C_G(\sigma)$ is the semidirect product $C_G(\sigma) = C_P(\sigma)C_S(\sigma)$ (see the proof of Theorem 2).

Now, consider the σ-fixed subgroup $C_{\widetilde{S}}(\sigma)$, and observe that $C_{\widetilde{S}}(\sigma)$ is the direct product $C_{\widetilde{S}}(\sigma) = C_S(\sigma) \times \langle\sigma\rangle$; indeed, σ centralizes $C_S(\sigma)$. Thus, by Theorem 2, $C_{\widetilde{S}}(\sigma)$ is an abelian p'-group with exponent dividing $q-1$ where $q = |\mathbb{k}^\sigma|$; moreover, it is clear that $C_{\widetilde{S}}(\sigma)$ acts on $\mathcal{J}$ as a group of $\mathbb{k}^\sigma$-linear ring automorphisms (here, $\mathcal{J}$ is naturally considered as a vector space over $\mathbb{k}^\sigma$). We note that the character $\widehat{\varphi} \in \mathrm{Irr}(P)$ is $C_{\widetilde{S}}(\sigma)$-invariant, and claim that $\widehat{\varphi} = \widehat{\tau}^P$ for some $C_{\widetilde{S}}(\sigma)$-invariant $\mathbb{k}^\sigma$-algebra subgroup Q of P and some $C_{\widetilde{S}}(\sigma)$-invariant linear character $\widehat{\tau}$ of Q; as in [6], a subgroup Q of P is said to be a $\mathbb{k}^\sigma$*-algebra subgroup* if $Q = 1 + \mathcal{U}$ for some $\mathbb{k}^\sigma$-subalgebra $\mathcal{U}$ of $\mathcal{J}$. To prove this, we proceed by induction on the dimension of $\mathcal{J}$. We consider the ($\mathbb{k}$-)algebra subgroup $N = 1 + \mathcal{J}^2$ of P; in fact, N is an ideal subgroup (and hence a normal subgroup) of P; an algebra subgroup of P is said to be an *ideal subgroup* if it is of the form $1 + \mathcal{I}$ for some (two-sided) ideal $\mathcal{I}$ of $\mathcal{J}$. Since $C_{\widetilde{S}}(\sigma)$ and P have coprime orders, [7, Theorem 13.27] asserts that there exists $\widehat{\eta} \in \mathrm{Irr}_{C_{\widetilde{S}}(\sigma)}(N)$ such that $\langle \widehat{\varphi}_N, \widehat{\eta}\rangle \neq 0$.

Firstly, assume that $\widehat{\eta}$ is not P-invariant. In this case, $I_P(\widehat{\eta})$ is a proper algebra subgroup of P (see [5, Lemma 3.3]); moreover, since $\widehat{\eta}$ is $C_{\widetilde{S}}(\sigma)$-invariant, $I_P(\widehat{\eta})$ is also $C_{\widetilde{S}}(\sigma)$-invariant. By [5, Lemma 3.2], there exists $\widehat{\varrho} \in \mathrm{Irr}_{C_{\widetilde{S}}(\sigma)}(I_P(\widehat{\eta}))$ such that $\langle \widehat{\varrho}, \widehat{\varphi}_N\rangle \neq 0$ and $\langle \widehat{\varrho}_N, \widehat{\eta}\rangle \neq 0$. By Clifford's correspondence (see [7, Theorem 6.11]), we must have $\widehat{\varphi} = \widehat{\varrho}^P$, and the claim follows by induction.

On the other hand, suppose that $\widehat{\eta}$ is P-invariant. In this case, we have $\widehat{\varphi}_N = e\widehat{\eta}$ for some positive integer e; moreover, [4, Theorem 1.3] asserts that $\widehat{\eta}$ is a linear character (and hence $e = \widehat{\varphi}(1)$). Let L be a $C_{\widetilde{S}}(\sigma)$-invariant $\mathbb{k}^\sigma$-algebra subgroup of P which is maximal with respect to the condition that $\widehat{\eta}$ is extendible to L. By [5, Lemma 3.2], there exists $\widehat{\tau} \in \mathrm{Irr}_{C_{\widetilde{S}}(\sigma)}(L)$ with $\langle \widehat{\tau}, \widehat{\varphi}_L\rangle \neq 0$ and $\langle \widehat{\tau}_N, \widehat{\eta}\rangle \neq 0$; since L/N is abelian, Gallager's theorem (see [7, Corollary 6.17] implies that $\widehat{\tau}_N = \widehat{\eta}$. We shall now prove that $\widehat{\varphi} = \widehat{\tau}^P$. To see this, we consider the inertia group $I_P(\widehat{\tau})$ and assume that $I_P(\widehat{\tau}) \neq L$. Let $\mathcal{I}$ and $\mathcal{I}'$ be the $\mathbb{k}^\sigma$-subalgebras of $\mathcal{J}$ such that $L = 1 + \mathcal{I}$ and $I_P(\widehat{\tau}) = 1 + \mathcal{I}'$; notice that $I_P(\widehat{\tau})$ is a $\mathbb{k}^\sigma$-algebra subgroup of P by [5, Lemma 3.3] (moreover, since $\mathcal{J}^2 \subseteq \mathcal{I}, \mathcal{I}'$, both $\mathcal{I}$ and $\mathcal{I}'$ are necessarily $\mathbb{k}^\sigma$-ideals of $\mathcal{J}$). Let $\mathbb{k}^\sigma[C_{\widetilde{S}}(\sigma)]$ denote the group algebra of $C_{\widetilde{S}}(\sigma)$ over the σ-fixed field $\mathbb{k}^\sigma$, and consider the left $\mathbb{k}^\sigma[C_{\widetilde{S}}(\sigma)]$-module $\mathcal{I}'/\mathcal{I}$. Let $\mathcal{V}$ be an irreducible $\mathbb{k}^\sigma[C_{\widetilde{S}}(\sigma)]$-submodule of $\mathcal{I}'/\mathcal{I}$; notice that we are assuming that $\mathcal{I}'/\mathcal{I}$ is non-zero. Since the exponent of $C_{\widetilde{S}}(\sigma)$ divides $q-1$ where $q = |\mathbb{k}^\sigma|$, $\mathbb{k}^\sigma$ is a splitting field for $C_{\widetilde{S}}(\sigma)$ (see [7, Corollary 9.25]), and thus $\mathcal{V}$ is one-dimensional (because $C_{\widetilde{S}}(\sigma)$ is abelian).

It follows that there exists $a \in \mathcal{J}' \setminus \mathcal{J}$ such that $\mathcal{J} + \Bbbk^\sigma a$ is an $C_{\widetilde{S}}(\sigma)$-invariant $\Bbbk^\sigma$-ideal of $\mathcal{J}$, and hence $L_a = 1 + \mathcal{J} + \Bbbk^\sigma a$ is an $C_{\widetilde{S}}(\sigma)$-invariant $\Bbbk^\sigma$-algebra subgroup of $1 + \mathcal{J}' = I_P(\widehat{\tau})$ such that $L \subseteq L_a$ and $|L_a : L| = q$. By [7, Theorem 13.28], there exists $\widehat{\tau}' \in \mathrm{Irr}_{C_{\widetilde{S}}(\sigma)}(L_a)$ such that $\langle \widehat{\tau}', \widehat{\tau}^{L_a} \rangle \neq 0$; hence, $\langle \widehat{\tau}'_L, \widehat{\tau} \rangle \neq 0$. By [6, Theorem A], both $\widehat{\tau}$ and $\widehat{\tau}'$ have q-power degree, and thus either $\widehat{\tau}'_L = \widehat{\tau}$ or $\widehat{\tau}' = \widehat{\tau}^{L_a}$. The first case cannot occur by the maximal choice of L. Therefore, $\widehat{\tau}' = \widehat{\tau}^{L_a}$, and thus $I_{L_a}(\widehat{\tau}) = L$ (by [7, Problem 6.1]). Since $L_a \subseteq I_P(\widehat{\tau})$, we conclude that $L_a \subseteq L$, a contradiction. It follows that $I_P(\widehat{\tau}) = L$, and this implies that $\widehat{\tau}^P \in \mathrm{Irr}(P)$ (by [7, Problem 6.1]). Since $\langle \widehat{\varphi}, \widehat{\tau}^P \rangle = \langle \widehat{\varphi}_L, \widehat{\tau} \rangle \neq 0$, we conclude that $\widehat{\varphi} = \widehat{\tau}^P$, as required.

Our claim is now proved; that is, there exist a $C_{\widetilde{S}}(\sigma)$-invariant $\Bbbk^\sigma$-algebra subgroup Q of P and a $C_{\widetilde{S}}(\sigma)$-invariant linear character $\widehat{\tau}$ of Q such that $\widehat{\varphi} = \widehat{\tau}^P$. In particular, Q is σ-invariant, and $\widehat{\tau} \in \mathrm{Irr}_\sigma(Q)$. Let $\tau = \pi_Q(\widehat{\tau}) \in \mathrm{Irr}(C_Q(\sigma))$; since $\widehat{\tau}$ is linear, it is clear that $\tau = \widehat{\tau}_{C_Q(\sigma)}$, and hence τ is linear and $C_{\widetilde{S}}(\sigma)$-invariant. By [1, Proposition 2.8], we conclude that $\varphi = \tau^{C_P(\sigma)}$; we recall that σ defines an $\Bbbk^\sigma$-linear automorphism of $\mathcal{J}$.

Finally, let $H = C_S(\sigma)Q$; we note that, since Q is $C_{\widetilde{S}}(\sigma)$-invariant (and $C_S(\sigma) \leq C_{\widetilde{S}}(\sigma)$), H is a subgroup of G satisfying $C_H(\sigma) = C_S(\sigma)C_Q(\sigma)$. Since τ is $C_{\widetilde{S}}(\sigma)$-invariant and $p \nmid |C_H(\sigma) : C_Q(\sigma)|$, [7, Corollary 6.28] implies that τ is extendible to $C_H(\sigma)$; in other words, there exists $\tau' \in \mathrm{Irr}(C_H(\sigma))$ such that $\tau'_{C_Q(\sigma)} = \tau$. Since $C_H(\sigma)/C_Q(\sigma)$ is abelian, we have

$$\tau^{C_H(\sigma)} = \sum_{\omega \in \mathrm{Irr}(C_H(\sigma)/C_Q(\sigma))} \omega\tau'$$

(by Gallagher's Theorem; see [7, Corollary 6.17]), and so

$$\varphi^{C_G(\sigma)} = (\tau^{C_P(\sigma)})^{C_G(\sigma)} = \tau^{C_G(\sigma)} = \sum_{\omega \in \mathrm{Irr}(C_H(\sigma)/C_Q(\sigma))} (\omega\tau')^{C_G(\sigma)}.$$

On the other hand, since $C_G(\sigma) = C_H(\sigma)C_P(\sigma)$ and $C_H(\sigma) \cap C_P(\sigma) = C_Q(\sigma)$, we deduce that

$$((\omega\tau')^{C_G(\sigma)})_{C_P(\sigma)} = ((\omega\tau')_{C_Q(\sigma)})^{C_P(\sigma)} = \tau^{C_P(\sigma)} = \varphi,$$

and thus $(\omega\tau')^{C_G(\sigma)}$ is irreducible for all $\omega \in \mathrm{Irr}(C_H(\sigma)/C_Q(\sigma))$. Since χ is an irreducible constituent of $\varphi^{C_G(\sigma)}$, we conclude that $\chi = (\omega\tau')^{C_G(\sigma)}$ for some $\omega \in \mathrm{Irr}(C_H(\sigma)/C_Q(\sigma))$, and this completes the proof of the theorem. $\square$

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