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ABSOLUTE REDUCTION OF BINARY FORMS

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Dedicated to the memory of Kay Magaard

ABSTRACT. Reduction theory of binary forms has been studied by Julia in [23] and more recently by several other authors. In this paper we introduce the absolute reduction and give an algorithm to compute the absolutely reduced form of any binary form. Such method can be applied to determine the minimal Weierstrass equation of a superelliptic curve over an integral ring.

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1. Introduction

Let $\mathcal{M}_g$ be the moduli space of genus $g \geq 2$ curves over an algebraically closed field F. For a moduli point $\mathfrak{p} \in \mathcal{M}_g$ we denote by K the minimal field of definition of $\mathfrak{p}$. It is a classical problem in algebraic geometry to find an equation of the curve $\mathcal{X}$ over K, corresponding to $\mathfrak{p}$. An algorithm to find such equations is known only for small genus g or for some classes of superelliptic curves (i.e. curves with affine equation $y^n = f(x)$). However such equations are not minimal, i.e., they do not have minimal height as defined in [32]. In this paper we introduce a method of finding a minimal equation for superelliptic curves defined over a ring of integers $\mathcal{O}_K$.

Any superelliptic curve with Weierstrass equation defined over the ring of integers $\mathcal{O}_K$ of a number field K is associated to a binary form $f(x, z)$ defined over $\mathcal{O}_K$. We associate to any binary form $f(x, z)$ a positive definite quadratic form $\mathcal{J}_f(x, z)$ called the Julia quadratic and therefore a point P_f in the upper half hyperbolic space $\mathcal{H}_3$. By an appropriate matrix $M \in \mathrm{SL}_2(\mathcal{O}_K)$ such a point is moved to a

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point P^M in the fundamental domain $\mathcal{F}_K$. This matrix moves f to a new binary form f^M, which is called the reduced form **red** (f) of f.

The form **red** (f) has small coefficients in its $\mathrm{SL}_2(\mathcal{O}_K)$-orbit, but it is not necessarily the form with the smallest height over $\mathcal{O}_K$. Therefore, it does not determine the superelliptic curve with the minimum height. Hence, we determine all twists $g_1, g_2, \ldots, g_r$ of **red** (f) with height less then or equal to the height of **red** (f), where r is the class number of the Julia quadratic $\mathcal{J}_f$.

We act on each of the twists by the transformations (ax, by) for certain a and b as explained in Thm. 9 to reduce the height even further. The minimal height among all the twists after such further reduction is called the *minimal absolute height*.

This paper is organized as follows. In the preliminaries we give some basics about fundamental domains. We start with the classical fundamental domain $\mathcal{F}$, which is obtain from the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane $\mathcal{H}_2$. Then, we describe the fundamental domain $\mathcal{F}_{\mathbb{Z}(i)}$ which is obtained by the action of the group $\Gamma_{\mathbb{Z}(i)} := \mathrm{SL}_2(\mathbb{Z}[i])/\{\pm I\}$ on the upper half space $\mathcal{H}_3$. Lastly we briefly discus fundamental domains of number fields, when such exists.

In Section 3 we start with giving some basic properties of binary forms and their invariants. Then, we define the height of binary forms and prove an equivalent of Northcot's theorem for binary form. See [32] for more details about this section.

In Section 4 we describe reduction theory of binary quadratics and binary quadratic Hermitian forms and then in Section 5 we describe reduction theory of higher degree binary forms. First, we explain in details the case of binary forms with real coefficients and then its generalization to binary forms with complex coefficients.

In Section 6 we explore some computational aspects of computing the Julia quadratic (invariant) and performing the reduction algorithm for higher degree binary forms. We give some geometric aspects of the reduction theory. Moreover, as we will see in Section 5 one of the key points of the reduction algorithm is computing the Jualia's quadratic. Expressing Julia's quadratic in terms of the covariants or the coefficients of the degree n binary form is only known for binary forms of degree 3, and 4. In Section 6 we provide a method how to compute the quadratic used for reduction for all possible signatures of binary forms with degree 5, and 6 in terms of the coefficients of the given binary form.

2. PRELIMINARIES

In this section we gice a brief review of what is well known in the literature, see [13,16,31,35] for more details. We start with the classical fundamental domain which we denote by $\mathcal{F}$ and is obtained from the action of the classical modular group on the upper half plane. Then we explore how one can generalize this notion when we go to three dimensional space and consider the action of a discrete subgroup of $\mathbb{C}$ in the upper half space. Lastly, in this section we generalize the concept of fundamental domain for any number field K, for more details see [13], [16].

The concept of the fundamental domain is crucial in developing the theory of reduction. Most of the theory in this chapter will be used in Section 4 and Section 5.

2.1. **Fundamental domain of** $\mathrm{SL}_2(\mathbb{Z})$**.** Let $\mathbb{P}^1$ be the Riemann sphere and $\mathrm{GL}_2(\mathbb{C})$ the group of 2×2 matrices with entries in $\mathbb{C}$. The group $\mathrm{GL}_2(\mathbb{C})$ acts on $\mathbb{P}^1$ by linear fractional transformations as follows

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} z = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{1}$$

where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ and $z \in \mathbb{P}^1$. It is easy to check that this is a group action. The $\mathrm{GL}_2(\mathbb{C})$ action on $\mathbb{P}^1$ is a transitive action, i.e. has only one orbit. Moreover, the action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{P}^1$ is also transitive.

For the rest of this section we will consider the action of $\mathrm{SL}_2(\mathbb{R})$ on the Riemann sphere. Notice that this action is not transitive. The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}^1$ has three orbits, namely $\mathbb{R} \cup \infty$, the upper half plane, and the lower-half plane. Therefore we restrict this action to the upper half-plane. Let $\mathcal{H}_2$ be the complex upper half plane, i.e.

$$\mathcal{H}_2 = \left\{ z = x + iy \in \mathbb{C} \;\middle|\; y > 0 \right\} \subset \mathbb{C}.$$

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}_2$ via linear fractional transformations. This action preserves $\mathcal{H}_2$ and acts transitively on it, further for $g \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathcal{H}_2$ we have

$$\mathrm{Im}(gz) = \frac{\mathrm{Im}\, z}{|\gamma z + \delta|^2}.$$

But $\mathrm{SL}_2(\mathbb{R})$ does not act faithfully on $\mathcal{H}_2$ since the elements $\pm I$ act trivially on $\mathcal{H}_2$. Hence, consider the above action as $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ action. This group acts faithfully on $\mathcal{H}_2$.

Let S be a set and G a group acting on it. Two points s_1, s_2 are said to be **G-equivalent** if $s_2 = g s_1$ for some $g \in G$. For any group G acting on a set S to itself we call a **fundamental domain** $\mathcal{F}$, if one exists, a subset of S such that any point in S is G-equivalent to some point in $\mathcal{F}$, and no two points in the interior of $\mathcal{F}$ are G-equivalent.

The group $\Gamma = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ is called the **modular group**. It is easy to prove that the Γ action on $\mathcal{H}_2$ via linear fractional transformations is a group action. This action has a fundamental domain $\mathcal{F}$

$$\mathcal{F} = \left\{ z \in \mathcal{H}_2 \;\middle|\; |z|^2 \geq 1 \text{ and } |Re(z)| \leq 1/2 \right\}$$

displayed in Fig. 1.

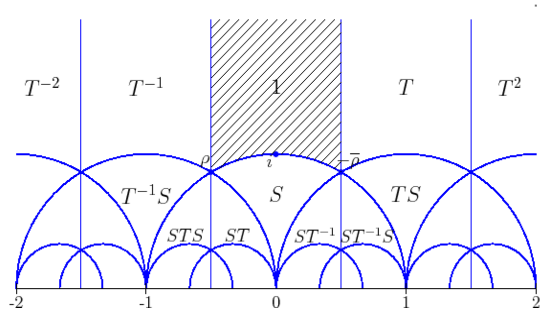

Figure 1. The action of the modular group on the upper half plane.

The following theorem proves that $\mathcal{F}$ is a fundamental domain, see [31] for proof.

Theorem 1. *i) Every $z \in \mathcal{H}_2$ is Γ-equivalent to a point in $\mathcal{F}$.*

ii) No two points in the interior of $\mathcal{F}$ are equivalent under Γ. If two distinct points z_1, z_2 of $\mathcal{F}$ are equivalent under Γ then $Re(z_1) = \pm 1/2$ and $z_1 = z_2 \pm 1$ or $|z_1| = 1$ and $z_2 = -1/z_1$.

iii) Let $z \in \mathcal{F}$ and $I(z) = \{g \,|\, g \in \Gamma,\, gz = z\}$ the stabilizer of $z \in \Gamma$. One has $I(z) = \{1\}$ except in the following cases:

$z = i$, in which case $I(z)$ is the group of order 2 generated by S;

$z = \rho = e^{2\pi i/3}$, in which case $I(z)$ is the group of order 3 generated by ST;

$z = -\overline{\rho} = e^{\pi i/3}$, in which case $I(z)$ is the group of order 3 generated by TS.

The canonical map $\mathcal{F} \to \mathcal{H}_2/\Gamma$ is surjective and its restriction to the interior of $\mathcal{F}$ is injective. The modular group Γ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where $S^2 = 1$ and $(ST)^3 = 1$. Note that $S^2 = 1$, so S has order 2, while $T^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ for any $k \in \mathbb{Z}$, so T has infinite order. For more details on the modular group and related arithmetic questions the reader can see [31] among others.

2.2. **Gaussian integers and the upper half space.** The upper half space $\mathcal{H}_3$ is defined as

$$\mathcal{H}_3 := \mathbb{C} \times (0, \infty) = \{(z,t) \,|\, z \in \mathbb{C}, t > 0\} = \{(x,y,t) | x, y \in \mathbb{R}, t > 0\}\,. \tag{2}$$

A point $P \in \mathcal{H}_3$ is given as $P = (z,t) = (x,y,t) = z + tj$, where $z = x + iy$ and $j = (0,0,1)$. The group $\mathrm{SL}_2(\mathbb{C})$ has a natural action on $\mathcal{H}_3$ by linear fractional transformations. Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ and $P = z + tj \in \mathcal{H}_3$. Then $P^M = z^* + t^* j \in \mathcal{H}_3$ where

$$z^* = \frac{(\alpha z + \beta)(\bar{\gamma}\bar{z} + \bar{\delta}) + \alpha\bar{\gamma}t^2}{||\gamma z + \delta||^2 + ||\gamma||^2 t^2} \quad \text{and} \quad t^* = \frac{t}{||\gamma z + \delta||^2 + ||\gamma||^2 t^2}.$$

The group $\mathrm{SL}_2(\mathbb{C})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where $a \in \mathbb{C}$. This generators act on (z,t), a point in $\mathcal{H}_3$, as follows

$$\begin{aligned} &\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : (z,t) \to (z + \alpha, t) \\ &\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : (z,t) \to \left(\frac{-\bar{z}}{|z|^2 + t^2}, \frac{t}{|z|^2 + t^2} \right). \end{aligned} \tag{3}$$

In analogy with the previous section we consider the action of a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ on $\mathcal{H}_3$. Let $\mathbb{Q}(i) \subset \mathbb{C}$ and $\mathbb{Z}[i]$ be the set of Gaussian integers. Then $\Gamma_{\mathbb{Z}(i)} := \mathrm{SL}_2(\mathbb{Z}[i])/\{\pm I\}$. A representation of $\Gamma_{\mathbb{Z}(i)}$ is given as follows

$$\Gamma_{\mathbb{Z}(i)} = \left\langle \; S, T, U, W \; \middle| \; \begin{array}{c} T^2 = U^2 = W^2 = 1 \\ (SW)^3 = (SU)^2 = (ST)^2 = 1 \\ (UW)^2 = (TW)^3 = 1 \end{array} \right\rangle$$

where

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

see [13, pg. 58-59] for more details. It is easy to prove that he group $\Gamma_{\mathbb{Z}[i]}$ is generated by S, T, U, W. The discrete group $\Gamma_{\mathbb{Z}[i]}$ acts on $\mathcal{H}_3$. Let $\mathcal{F}_{\mathbb{Z}(i)}$ be the following

$$\mathcal{F}_{\mathbb{Z}(i)} = \left\{ (z,t) \,|\, z = x+iy, \, -\frac{1}{2} \leq x \leq \frac{1}{2}, 0 \leq y \leq \frac{1}{2}, ||z||^2 + t^2 \geq 1 \right\}. \tag{4}$$

Given a point $\omega \in \mathcal{H}_3$ there exists $M \in \Gamma_{\mathbb{Z}[i]}$ such that $\omega^M \in \mathcal{F}_{\mathbb{Z}(i)}$. Moreover, if we suppose ω, and ω' are in the same Γ-orbit such that $\omega' = M\omega$ for $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_{\mathbb{Z}[i]}$. Assume $t(\omega) \leq t(\omega')$. Then we have one of the following three cases

i) $\gamma = 0$.

ii) $||\gamma|| = 1$, $t^2 \leq 1$.

iii) $||\gamma||^2 = 2$, $t^2 = 1/2$, ω is in the boundary of $\mathcal{F}_{\mathbb{Z}(i)}$, $\gamma z + \delta = 0$, $\delta = \pm 1, \pm i$.

Hence, from all the above we can conclude that $\mathcal{F}_{\mathbb{Z}(i)}$ is a fundamental domain of action of $\Gamma_{\mathbb{Z}[i]}$ on $\mathcal{H}_3$. Graphically $\mathcal{F}_{\mathbb{Z}(i)}$ is presented in Fig. 2.

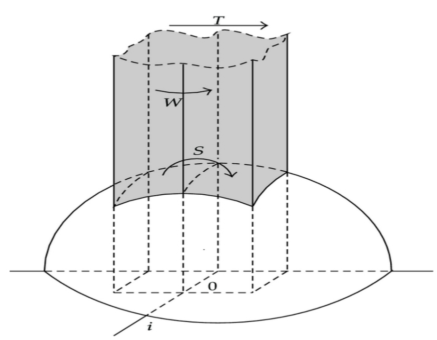

FIGURE 2. The fundamental domain $\mathcal{F}_{\mathbb{Z}(i)}$ in the upper half space

2.3. **Other algebraic number fields.** In this section we describe fundamental domains of other algebraic number fields. The action described in Eq. (1) makes sense when $\mathbb{C}$ is replaced by any number field K.

For analogy of $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{R})$ we need to consider a discrete subring of $\mathbb{C}$. For any number field K with $\mathcal{O}_K$ its ring of integers the natural thing to consider is $\mathrm{SL}_2(\mathcal{O}_K)$, which is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$. In an analogues way we can prove that the generators of $\mathrm{SL}_2(\mathcal{O}_K)$ are $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for $a \in \mathcal{O}_K$. Next we want to consider for which number fields K the group $\mathrm{SL}_2(\mathcal{O}_K)$ acts transitively on $\mathbb{P}^1(K)$.

Let us recall some basic definitions from number theory, [30]. A **fractional ideal** is an $\mathcal{O}_K$-submodule $\mathfrak{a}$ contained in K such that there exists an element $c \neq 0$ in $\mathcal{O}_K$ satisfying $c\mathfrak{a} \subset \mathcal{O}_K$. Let $\mathfrak{P}$ be the subset of fractional ideals, then we write $\mathfrak{a} \sim \mathfrak{b}$ if there exists an element $\lambda \in K^*$ such that $\mathfrak{a} = (\lambda)\mathfrak{b}$, i.e. $\mathfrak{a}\mathfrak{b}^{-1}$ is a principal fractional ideal. The equivalence classes of fractional ideals form a finite group which we call the **ideal class group**. Its order is usually denoted by h_K and is called the **class number** of K. In [5,31], amongst others, it is proved that for a number field K, the number of orbits for $\mathrm{SL}_2(\mathcal{O}_K)$ on $\mathbb{P}^1(K)$ is the class number of K.

Hence, there is a bijection between the set of orbits of $\mathrm{SL}_2(\mathcal{O}_K)$ on $\mathbb{P}^1(K)$ and the ideal class group of K. Moreover, $\mathrm{SL}_2(\mathcal{O}_K)$ acts transitively on $\mathbb{P}^1(K)$ if and only if K has class number 1.

Next we see how these results apply to imaginary quadratic number fields. Let $K = \mathbb{Q}(\sqrt{\Delta}) \subset \mathbb{C}$ be an imaginary quadratic number field where $\Delta < 0$ a square-free integer, d_K the discriminant of K, and $\mathcal{O}_K$ its ring of integers. The group $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K)$ is called the "**Bianchi group**" and is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

It is easy to show that the Bianchi group acts on $\mathcal{H}_3$. This action has a fundamental domain, which we will denote as $\mathcal{F}_K$ and depends on K. For small discriminant this was determined by Bianchi and others in the 19^{th} century.

Consider the $\mathrm{PSL}_2(\mathcal{O}_K)$ action on $\mathcal{H}_3$, and define the following:

$$
\begin{aligned}
\mathcal{B}_K &= \left\{ z + rj \in \mathcal{H}_3 \;\middle|\; |cz+d|^2 + |d|^2 r^2 \geq 1, \text{ for all } c, d \in \mathcal{O}_K : \langle c, d\rangle = \mathcal{O}_K \right\} \\
\mathcal{P}_K &= \left\{ z \in \mathbb{C} \;\middle|\; 0 \leq \mathrm{Re}(z) \leq 1, \quad 0 \leq \mathrm{Im}(z) \leq \sqrt{|d_K|}/2 \right\} \\
F_K &= \mathcal{P}_K, \text{ for } \Delta \neq -3, -1 \\
F_{\mathbb{Q}(i)} &= \left\{ z \in \mathbb{C} \;\middle|\; 0 \leq |\mathrm{Re}(z)| \leq \frac{1}{2}, \; 0 \leq \mathrm{Im}(z) \leq \frac{1}{2} \right\} \\
F_{\mathbb{Q}(\sqrt{-3})} &= \left\{ z \in \mathbb{C} \;\middle|\; 0 \leq \mathrm{Re}(z), \; \frac{\sqrt{3}}{3}\mathrm{Re}(z) \leq \mathrm{Im}(z), \, \mathrm{Im}(z) \leq \frac{\sqrt{3}}{3}\Big(1 - \mathrm{Re}(z)\Big) \right\} \\
&\cup \left\{ z \in \mathbb{C} \;\middle|\; 0 \leq \mathrm{Re}(z) \leq \frac{1}{2}, \; -\frac{\sqrt{3}}{3}\mathrm{Re}(z) \leq \mathrm{Im}(z) \leq \frac{\sqrt{3}}{3}\mathrm{Re}(z) \right\} \\
\mathcal{F}_K &= \left\{ z + rj \in \mathcal{B}_K \;\middle|\; z \in F_K \right\}.
\end{aligned}
$$

Then the following theorem is true and see [16, pg 319] for the proof.

Theorem 2. *The set $\mathcal{F}_K$ is a fundamental domain for* $\mathrm{PSL}_2(\mathcal{O}_K)$.

Assume $(z, r) \in \mathcal{F}_K$, from the definition of the fundamental domain $\mathcal{F}_K$ we get obvious bounds for z. The following proposition gives a lower bound on r. The proof can be found in [16, pg. 316].

Proposition 1. *There is a constant $k \in \mathbb{R}^{>0}$ only depending on the number field K so that for any $z \in \mathbb{C} \setminus K$ there are infinitely many $\lambda, \mu \in \mathcal{O}_K$ with*

$$
\left| z - \frac{\lambda}{\mu} \right| \leq \frac{k}{|\mu|^2}
$$

and $\langle \lambda, \mu \rangle = \mathcal{O}_K$.

Hence for big enough μ we have $\frac{k}{|\mu|^2} < 1$ and therefore $\left| z - \frac{\lambda}{\mu} \right| < 1$. But from the definition of $\mathcal{B}_K$, as given above, for all $\lambda, \mu \in \mathcal{O}_K$ such that $\langle \lambda, \mu \rangle = \mathcal{O}_K$ we have $|\mu z - \lambda|^2 + |\mu|^2 r^2 \geq 1$, and we can conclude that $r \geq r_K$, for some r_K depending on the number field K. Consider the set

$$
S_K = \left\{ z \in K \mid \; |z\mu + \lambda| \geq 1 \text{ for all } \langle \lambda, \mu \rangle = \mathcal{O}_K \right\}.
$$

This is the set of singular points. In [16] it is proved that $z+rj \in \mathcal{F}_K$ for $z \in S_K$ are the only points in the fundamental domain such that r is not bounded from below. But when the number field K has class number one this set is empty. Hence, for an imaginary number field K, $h_K = 1$, there exists a constant r_K, only depending on K, such that $r \geq r_K$ for every $(z, r) \in \mathcal{F}_K$.

In [29] it is shown that when $K = \mathbb{Q}(\sqrt{-D})$ and D is one of 1, 2, ,3, 7, 11, 19, 43, 67, 163, then the value of r_K^2 is as given in Table Table 1.

TABLE 1. The value of r_K^2 for some number fields K

D	1	2	3	7	11	19	43	67	163
r_K^2	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{7}$	$\frac{2}{11}$	$\frac{2}{19}$	$\frac{2}{43}$	$\frac{2}{67}$	$\frac{2}{163}$

This will be used in the following sections when we will introduce reduction theory of binary quadratics, as well as degree n binary forms. We can get bounds on the coefficients of a binary form depending only on the number field K, c.f. Section 4.

Lastly, let K be a number field. K is called **totally real** if for each embedding of K into the complex numbers the image lies inside the real numbers. Equivalently, K is generated over $\mathbb{Q}$ by one root of an integer polynomial P, all of the roots of P are real. If K is a totally real algebraic number field the group

$$\Gamma_K = \mathrm{PSL}_2(\mathcal{O}_K) = \mathrm{SL}_2(\mathcal{O}_K)/\{\pm I\}$$

is called the **Hilbert modular group** of K. If $[K : \mathbb{Q}] = n$ then the n-embeddings of K into $\mathbb{R}$ define an embedding of $\mathrm{PSL}_2(K)$ into $\mathrm{PSL}_2(\mathbb{R})^n$. When $n = 1$, we have the classical modular group described in Section 2.1.

The group Γ_K acts properly discontinuously on $\mathcal{H}^n$ which is contained in $\mathbb{P}^1(\mathbb{C}) \times \cdots \times \mathbb{P}^1(\mathbb{C})$, n-times. This generalizes the well known action of the classical modular group on the upper-half space $\mathcal{H}$. The orbits of $\mathbb{P}^1(K)$ under Γ_K or any group $\Gamma \subset \mathrm{PGL}_2(K)^+$ which is discrete in $\mathrm{PSL}_2(\mathbb{R})^+$ are called the **cusps** of Γ_K or Γ. For more details see [35].

3. HEIGHTS OF BINARY FORMS

In this section we give some of the basic properties of the binary forms and their invariants. We also define the height of a binary form, see [15, 19, 24, 32, 33] and others for more details.

Throughout this section k is an algebraically closed field of characteristic zero. Let $k[x, y]_n$ be the space of degree $n \geq 2$ homogenous polynomials. The group $k^\star$ acts on $k[x, y]_n$ by multiplication by a constant. The space of degree n binary forms with coefficients from k will be denoted by $V_{n,k} := k[x, y]_n / k^\star$. Thus, by a binary form $f \in V_{n,k}$ we will always mean the equivalence class of f.

The group $SL_2(k)$ acts on $V_{n,k}$ in the usual way. For an element $g \in V_{n,k}$ and $M \in SL_2(k)$, the action of M on g will be denoted by g^M. Two binary forms f and g are called *k-equivalent* if there is $M \in SL_2(k)$ such that $f = g^M$.

3.1. Invariants and covariants.

Let $A_0, A_1, \ldots, A_n$ be coordinate functions on $V_{n,k}$. Then the coordinate ring of $V_{n,k}$ can be identified with $k[A_0, \ldots, A_n]$. For $I \in k[A_0, \ldots, A_n]$ and $M \in GL_2(k)$, define $I^M \in k[A_0, \ldots, A_n]$ as follows

$$I^M(f) = I(f^M) \tag{5}$$

for all $f \in V_{n,k}$. Then $I^{MN} = (I^M)^N$ and Eq. (5) defines an action of $GL_2(k)$ on $k[A_0, \ldots, A_n]$.

Definition 1. *Let $\mathcal{R}_n$ be the ring of $SL_2(k)$ invariants in $k[A_0, \ldots, A_n]$, i.e., the ring of all $I \in k[A_0, \ldots, A_n]$ with $I^M = I$ for all $M \in SL_2(k)$.*

A homogeneous polynomial $I \in k[A_0, \ldots, A_n, x, y]$ is called a **covariant** of index s if

$$I^M(f) = \delta^s I(f),$$

where $\delta = \det(M)$. The homogeneous degree in $A_1, \ldots, A_n$ is called the **degree** of I, and the homogeneous degree in x, y is called the **order** of I. A covariant of order zero is called an **invariant**. An invariant is a $SL_2(k)$-invariant on V_n. The discriminant of a binary form $f \in V_{n,k}$ is an $SL_2(k)$-invariant or order $2n-2$ which is denoted by Δ_f.

Since k is algebraically closed, any binary form $f(x,y)$ can be factored as

$$f(x,y) = (y_1 x - x_1 y) \cdots (y_d x - x_d y) = \prod_{1 \le i \le d} \det \begin{pmatrix} x & x_i \\ y & y_i \end{pmatrix} \tag{6}$$

The points with homogeneous coordinates $(x_i, y_i) \in \mathbb{P}^1$ are called the roots of the binary form f. Thus, for $M \in GL_2(k)$ we have

$$g\left(f(x,y)\right) = (\det(M))^d \cdot (y_1' x - x_1' y) \cdots (y_d' x - x_d' y),$$

where

$$\begin{pmatrix} x_i' \\ y_i' \end{pmatrix} = g^{-1} \begin{pmatrix} x_i \\ y_i \end{pmatrix}. \tag{7}$$

Now we define the height and the minimal height of a binary form. An extended overview of this section can be found in [32]. Let K be an algebraic number field, M_K denotes the set of valuations of K, and $f \in K[x,y]$ a degree n binary form given by

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_{n_1} x y^{n-1} + a_n y^n$$

The **affine height** of f is defined as follows

$$H_K^{\mathbb{A}}(f) = \prod_{v \in M_K} \max \left\{ 1, |f|_v^{n_v} \right\}$$

where

$$|f|_v := \max_j \left\{ |a_j|_v \right\}$$

is the **Gauss norm** for any absolute value v. The **projective height** of a polynomial is the height of its coefficients taken as coordinates in the projective space. Thus,

$$H_K(f) = \prod_{v \in M_K} |f|_v^{n_v}$$

From now on, when we say "height" of a binary form f we will always mean the projective height $H_K(f)$. If $K = \mathbb{Q}$ then we will just use $H(f)$. The **(projective) absolute multiplicative height** is defined as follows

$$\begin{aligned}\mathrm{H} : \mathbb{P}^n(\mathbb{Q}) &\to [1, \infty) \\ \mathrm{H}(f) &= \mathrm{H}_K(f)^{1/[K:\mathbb{Q}]},\end{aligned}$$

and in the same way $h(f)$, $\mathrm{H}^{\mathbb{A}}(f), h^{\mathbb{A}}(f)$. In [32] the authors prove the following.

Theorem 3. *Given $F(x, y) \in K[x, y]$. There are only finitely many polynomials $G(x, y) \in K[x, y]$ such that $\mathrm{H}_K(G) \leq \mathrm{H}_K(F)$.*

Let $f \in V_{n,\mathbb{C}}$. If there exists a matrix $M \in GL_2(\mathbb{C})$ such that $f^M \in \mathcal{O}_K[x, y]$ for some number field K, we say that f has an **integral model over** K. If f has an integral model over $\mathbb{Q}$ we simply say that f has an integral model. The main goal of this paper is to determine the integral model of a binary form f with minimal height when such model exists.

4. Reduction theory of binary quadratics

4.1. **Binary quadratic forms over $\mathbb{R}$.** First we present some basics about binary quadratic forms. Let $Q(X, Z) = aX^2 + bXZ + cZ^2$ be a binary quadratic in $\mathbb{R}[X, Z]$. We will use the following notation to represent the equivalence class of binary quadratics up to a scalar multiple, $Q(X, Z) = [a, b, c]$. The **discriminant** of Q is $\Delta = b^2 - 4ac$ and $Q(X, Z)$ is positive definite if $a > 0$ and $\Delta < 0$. Denote the set of positive definite binary quadratics with $V^+_{2,\mathbb{R}}$, i.e.

$$V^+_{2,\mathbb{R}} = \left\{ Q(X, Z) \in \mathbb{R}[X, Z] \;\middle|\; Q(X, Z) \text{ is positive definite} \right\}.$$

Let $\mathrm{SL}_2(\mathbb{R})$ act as usual on the set of positive definite binary quadratic forms

$$\begin{aligned}\mathrm{SL}_2(\mathbb{R}) \times V^+_{2,\mathbb{R}} &\to V^+_{2,\mathbb{R}} \\ \begin{pmatrix}\alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4\end{pmatrix} \times \begin{pmatrix} X \\ Z \end{pmatrix} &\to Q(\alpha_1 X + \alpha_2 Z, \alpha_3 X + \alpha_4 Z)\end{aligned}$$

We will denote this new form with $Q^M(X, Z) = a'X^2 + b'XZ + c'Z^2$ where

$$\begin{aligned}a' &= a\alpha_1^2 + b\alpha_1\alpha_3 + c\alpha_3^2 \\ b' &= 2(a\alpha_1\alpha_2 + c\alpha_3\alpha_4) + b(\alpha_1\alpha_4 + \alpha_2\alpha_3) \\ c' &= a\alpha_2^2 + b\alpha_2\alpha_4 + c\alpha_4^2\end{aligned} \tag{8}$$

and

$$\Delta' = b'^2 - 4a'c' = (\det M)^2 \Delta.$$

Obviously Δ is fixed under the $\mathrm{SL}_2(\mathbb{R})$ action and the leading coefficient of the new form Q^M will be $Q^M(1, 0) = Q(a, c) > 0$. Hence, $V^+_{2,\mathbb{R}}$ is preserved under this action.

Now, consider the following map which is called the **zero map**

$$\begin{aligned}\xi : V^+_{2,\mathbb{R}} &\to \mathcal{H}_2 \\ [a, b, c] &\mapsto \xi(Q) = \frac{-b + \sqrt{\Delta}}{2a}\end{aligned} \tag{9}$$

where $\mathrm{Re}(\xi(Q)) = -\frac{b}{2a}$, and $\mathrm{Im}(\xi(Q)) = \frac{\sqrt{|\Delta|}}{2a}$. This map is a bijection since given $z = x + iy$, we can find a, b, c such that $Q(X, Z)$ is positive definite given as $[1, -2x, x^2 + y^2]$.

Note that this map gives us a one-to-one correspondence between positive definite quadratic forms and points in $\mathcal{H}_2$. Let Γ be the modular group acting on $\mathcal{H}_2$, and on $V^+_{2,\mathbb{R}}$ as described above. Then the following theorem is proved in [5].

Lemma 1. *The zero map $\xi : V^+_{2,\mathbb{R}} \to \mathcal{H}_2$ is a Γ-equivariant map. In other words, $\xi(Q^M) = M^{-1}\xi(Q)$.*

4.2. **Reduction theory for binary quadratics.** We denoted with $V^+_{2,\mathbb{R}}$ the set of positive definite quadratics and we have defined an equivalence relation in this set. Define $Q = [a, b, c]$ to be **reduced** if $\xi(Q) \in \mathcal{F}$. Moreover, it is easy to prove that a positive definite quadratic form $Q \in V^+_{2,\mathbb{R}}$ is reduced if and only if $|b| \leq a \leq c$. This gives an arithmetic condition on the coefficients of a reduced positive definite binary quadratic.

Note that if Q is a reduced form with fixed discriminant $\Delta = -D$, then $b \leq \sqrt{D/3}$. Moreover, the number of reduced forms of a fixed discriminant $\Delta = -D$ is finite. Every positive definite quadratic form Q with fixed discriminant is equivalent to a reduced form of the same discriminant. Two reduced binary quadratics are equivalent only in the following two cases $[a, b, a] \sim [a, -b, a]$, and $[a, a, c] \sim [a, -a, c]$. The proof of this fact can be found in [12, pg. 15]. Let $\Delta < 0$ be fixed, then the class number $h(\Delta)$ is equal to the number of primitive reduced forms of discriminant Δ.

In [5] the authors give an algorithm to list reduced forms with given discriminant. In [5, Table 1] the authors list (count the number of) reduced forms with fixed discriminant $\Delta \equiv 1 \bmod 4$, $\Delta \leq 0$. Note that n represents the number of reduced forms with discriminant Δ.

From the equivalence classes of reduced quadratics there is one which has the smallest height. We call this class the special class and the corresponding height the minimal absolute height. Being able to construct such tables has two benefits. First we can count the equivalence classes and second we can find the quadratics with minimal height in their respective orbits.

The following theorem gives a connection between the concept of a reduced form and the height of the $\mathrm{SL}_2(\mathbb{Z})$-equivalence class $[f]$ of a binary quadratic form f.

Theorem 4. *Let $f(X, Z) = aX^2 + bXZ + cZ^2$ be reduced (i.e. $|b| < a < c$). Then $\mathrm{H}([f]) = c$.*

Proof. We want to show that given any $M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & a_4 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ acting on $f(X, Z)$ we have that $\max\{|a_1|, |b_1|, |c_1|\} \geq c$, where a_1, b_1, c_1 are the coefficients of the new form f^M. From Eq. (8) we have

$$\begin{aligned} a_1 &= a\alpha_1^2 + b\alpha_1\alpha_3 + c\alpha_3^2 \\ b_1 &= 2(a\alpha_1\alpha_2 + c\alpha_3\alpha_4) + b(\alpha_1\alpha_4 + \alpha_2\alpha_3) \\ c_1 &= a\alpha_2^2 + b\alpha_2\alpha_4 + c\alpha_4^2. \end{aligned}$$

We will prove it only for the generators of $\mathrm{SL}_2(\mathbb{Z})$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. First, let $M = S$, then we have $[a_1, b_1, c_1] = [c, -b, a]$ and if $M = T$ then $[a_1, b_1, c_1] = [a, 2a + b, a + b + c]$ and the result is obvious. $\square$

Corollary 1. *If f is a reduced quadratic then f has minimal height* $\mathrm{H}(f)$ *in its* Γ*-orbit.*

Proof. Above it is proved that $f(x, y) = ax^2 + bxy + cy^2$ being reduced is equivalent to $|b| \leq a \leq c$. Moreover, $\mathrm{H}(f) = c$. This shows that f has minimal height in its Γ-orbit. $\square$

Next we focus on binary Hermitian forms and then we can generalize the reduction theory for number fields, when possible.

4.3. **Binary Hermitian forms.** In this section first we give some basics from linear algebra about Hermitian matrices and Hermitian binary forms. Then we describe the $PSL_2(\mathbb{C})$ action on the 3-dimensional hyperbolic space, denoted by $\mathcal{H}_3$ and define the "zero" map which gives a one-to-one correspondence between positive definite Hermitian forms and points in $\mathcal{H}_3$. At the end of the section we will define reduction of Hermitian forms and give an algorithm how to perform reduction.

An $n \times n$ matrix A with complex entries is called Hermitian if $A^* = A$, where $A^* = \bar{A}^T$. Recall that $\bar{A}$ is obtained from A by applying complex conjugation to all elements and A^T is the transpose of A. By the definition we see that an Hermitian matrix is unchanged by taking its conjugate transpose. Note that any Hermitian matrix must have real diagonal entries.

Let R be a subring of $\mathbb{C}$ with $R = \bar{R}$, denote by $H(R)$ the set of 2×2 Hermitian matrices, i.e.

$$H(R) = \{A \in M_2(R) \,|\, A^* = A\}$$

A 2×2 matrix is in $H(R)$ if it is of the form $A = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$ where $a, d \in R \cap \mathbb{R}$ and $b \in R$. Every matrix $A \in H(R)$ defines a **binary Hermitian form** with entries in R. If $A \in H(R)$ then the associated binary Hermitian form is the semi-quadratic map

$$Q : \mathbb{C} \times \mathbb{C} \to R$$

defined as

$$Q(X, Z) = \begin{pmatrix} X \\ Z \end{pmatrix}^\star \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \begin{pmatrix} X \\ Z \end{pmatrix} = aX\bar{X} + \bar{b}X\bar{Z} + b\bar{X}Z + dZ\bar{Z}.$$

The discriminant $\Delta(Q)$ of $Q \in H(R)$ is defined as $\Delta(Q) = \det\,(Q) = ad - |b|^2$. A binary Hermitian form $Q \in H(R)$ is **positive definite** if $Q(X, Z) > 0$ for every $(X, Z) \in \mathbb{C} \times \mathbb{C} \setminus \{0, 0\}$. Q is called **negative definite** if $-Q$ is positive definite and **indefinite** if $\Delta(Q) < 0$. Denote by $H(R)^+$ the set of positive definite Hermitian forms, i.e.

$$H(R)^+ = \{Q \in H(R) \,|\, Q \textit{ is positive definite}\}$$

If $a \neq 0$, then

$$Q(X, Z) = a\left(\left|X + \frac{bZ}{a}\right|^2 + \frac{\Delta}{a^2}|Z|^2\right).$$

Hence, $Q \in H^+(R)$ if and only if $a > 0$ and $\Delta > 0$. The group $\mathrm{GL}_2(R)$, where $R \subset \mathbb{C}$, as in Section 4.3, acts on $H(R)$ as follows

$$\begin{aligned} \mathrm{GL}_2(R) \times H(R) &\to H(R) \\ (M, Q) &\mapsto M^\star Q M \end{aligned} \tag{10}$$

for $M \in \mathrm{GL}_2(R)$ and $Q \in H(R)$. We can define in an analogous way an $\mathrm{SL}_2(R)$-action on $H(R)$. Note that if A is the Hermitian matrix of Q then the Hermitian matrix of the new form is $M^\star AM$. It is easy to show that

$$\Delta(M(Q)) = |\det M|^2 \cdot \Delta(Q). \tag{11}$$

The group $\mathrm{GL}_2(R)$ leaves $H^+(R)$ invariant since for $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $Q \in H^+(R)$, from Eq. (11) we have that $\Delta(M(Q)) > 0$ and also it is easy to check that the leading coefficient of $Q^M = Q(\alpha, \gamma) > 0$.

The group $\mathbb{R}^{>0}$ acts on $H^+(\mathbb{C})$ by scalar multiplication. We will denote by $\tilde{H}^+(\mathbb{C})$ the quotient space $H^+(\mathbb{C})/\mathbb{R}^{>0}$, and $[Q]$ the equivalence class of Q in $\tilde{H}^+(\mathbb{C})$. The action of $\mathrm{GL}_2(\mathbb{C})$ on $H(\mathbb{C})$ induces an action of $\mathrm{GL}_2(\mathbb{C})$ on $\tilde{H}^+(\mathbb{C})$.

The center of $\mathrm{SL}_2(\mathbb{C})$ acts trivially on $H(\mathbb{C})$, so we get an induced action of $\mathrm{PSL}_2(\mathbb{C})$ on $H(\mathbb{C})$ and $\tilde{H}^+(\mathbb{C})$.

Definition 2. *The map $\xi : H^+(\mathbb{C}) \to \mathcal{H}_3$ defined by*

$$\xi \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \to -\frac{b}{a} + \frac{\sqrt{\Delta(Q)}}{a} \cdot j \tag{12}$$

*is called the "**zero map**" for binary quadratic Hermitian forms. Clearly ξ induces a map $\xi : \tilde{H}^+(\mathbb{C}) \to \mathcal{H}_3$.*

Since Q is positive definite we have that $a > 0$ and $\Delta > 0$, hence ξ is well defined and continuous. This map is a bijection since given $(z, t) \in \mathcal{H}_3$ we can find $Q = [1, -z, -\bar{z}, |z|^2 + t^2]$, i.e.

$$Q : (u, v) \to |u|^2 - zu\bar{v} - \bar{z}\bar{u}v + (|z|^2 + t^2)|v|^2$$

Therefore, this map gives a one-to-one correspondence between equivalence classes of positive definite binary quadratic Hermitian forms and points in $\mathcal{H}_3$. The following theorem holds.

Theorem 5. *The map $\xi : \tilde{H}^+(\mathbb{C}) \to \mathcal{H}_3$ defined by*

$$[Q] \to -\frac{b}{a} + \frac{\sqrt{\Delta(Q)}}{a} \cdot j$$

is a $\mathrm{PSL}_2(\mathbb{C})$ equivariant, i.e. ξ satisfies $\xi(Q^M) = M^{-1}\xi(Q)$ for every $M \in \mathrm{PSL}_2(\mathbb{C})$ and $Q \in \mathcal{H}^+(\mathbb{C})$.

Proof. See [5].

Note that Thm. 5 holds if we replace $\mathbb{C}$ by any number field K and the proof follows through in exactly the same way.

4.4. **Reduction theory of Hermitian forms.** Reduction of real binary quadratic forms with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$, as described in Section 4.2, may be extended to a reduction theory for binary forms with complex coefficients (Hermitian binary forms) under the action of certain discrete subgroups of $\mathbb{C}$. In order to do that we need a discrete subring of $\mathbb{C}$ and then define the fundamental domain of this action.

Let $H(\mathcal{O}_K)$ denote the space of binary Hermitian forms with coefficients in $\mathcal{O}_K$, and by $H^+(\mathcal{O}_K)$ denote the set of positive definite Hermitian forms with coefficients in $\mathcal{O}_K$, and let $H^-(\mathcal{O}_K)$ the set of indefinite Hermitian forms with coefficients in

$\mathcal{O}_K$. It is easy to show that the **"Bianchi group"** $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K)$ acts on $H^+(\mathcal{O}_K)$ preserving discriminants.

The following definition is analog to the one for positive definite binary quadratic forms.

Definition 3. *A positive definite Hermitian form $f \in H^+(\mathcal{O}_K)$ is called a* ***reduced Hermitian form*** *if $\xi(f) \in \mathcal{F}_K$.*

Let K be an imaginary quadratic number field and $\mathcal{O}_K$ its ring of integers. Define

$$H(\mathcal{O}_K, \Delta) = \{f \in H(\mathcal{O}_K) \,|\, \Delta(f) = \Delta\}$$

to be the subspace of $H(\mathcal{O}_K)$ with fixed discriminant Δ and

$$H^{\pm}(\mathcal{O}_K, \Delta) = \{f \in H^{\pm}(\mathcal{O}_K) \,|\, \Delta(f) = \Delta\}$$

the subspace of $H^{\pm}(\mathcal{O}_K)$ of fixed discriminant. Then the following theorem holds.

Theorem 6. *Given $\Delta \neq 0 \in \mathbb{Z}$, the number of reduced forms of $H(\mathcal{O}_K, \Delta)$ is finite.*

The proof can be found in [16, pg. 411].

Corollary 2. *For any $\Delta \in \mathbb{Z}$ with $\Delta \neq 0$ the set $H(\mathcal{O}_K, \Delta)$ (and $H^{\pm}(\mathcal{O}_K, \Delta)$) splits into finitely many $\mathrm{SL}_2(\mathcal{O}_K)$-orbits.*

Proof. This is an immediate consequence of Thm. 8 and Thm. 5 which says that every $f \in H(\mathcal{O}_K, \Delta)$ is $\mathrm{PSL}_2(\mathcal{O}_K)$-equivalent to a reduced form. □

For any $\Delta \in \mathbb{Z}$ with $\Delta \neq 0$ define

$$\tilde{H}(\mathcal{O}_K, \Delta) = \mathrm{SL}_2(\mathcal{O}_K) \backslash H(\mathcal{O}_K, \Delta),$$

and denote by $h(\mathcal{O}_K, \Delta) := \left| \tilde{H}(\mathcal{O}_K, \Delta) \right|$, where the number $h(\mathcal{O}_K, \Delta)$ is called the **class number of binary Hermitian forms of discriminant** Δ.

We define in the same way for positive definite Hermitian forms $\tilde{H}^+(\mathcal{O}_K, \Delta) = \mathrm{SL}_2(\mathcal{O}_K) \backslash H^+(\mathcal{O}_K, \Delta)$ such that $h^+(\mathcal{O}_K, \Delta) = \left| \tilde{H}^+(\mathcal{O}_K, \Delta) \right|$, and $h^+(\mathcal{O}_K, \Delta)$ is called the **class number of positive definite binary Hermitian forms of discriminant** Δ. Note that for $\Delta > 0$ we have that $h(\mathcal{O}_K, \Delta) = 2h^+(\mathcal{O}_K, \Delta)$.

Given $\mathcal{O}_K$ and the discriminant Δ it is always possible to compute the class number of positive definite binary Hermitian forms with given discriminant Δ. For a reduced binary Hermitian form we can get bounds on the coefficients of the form depending only on the number field K.

Let $Q(X, Z) = aX\bar{X} + \bar{b}X\bar{Z} + b\bar{X}Z + cZ\bar{Z}$ be a reduced Hermitian form, with discriminant Δ and let $D = |\Delta|$. We have

$$a \leq \frac{\sqrt{D}}{r_k}, \;\; |b|^2 \leq c_k a^2, \;\; \text{and } ac \leq \left(1 + \frac{c_k}{r_k}\right) D$$

for constant c_k depending only on the number field K.

Let us now consider the case when $K = \mathbb{Q}(i)$. The fundamental domain of this action is $\mathcal{F}_{\mathbb{Z}(i)}$, as shown in Eq.(4). We want to count the number of reduced positive definite binary Hermitian forms with a fixed discriminant Δ, i.e. $h^+(\mathbb{Z}[i], \Delta)$.

Let $f = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ be a positive definite binary quadratic Hermitian form with coefficients in $Z[i]$ and non-zero discriminant Δ. The binary quadratic Hermitian form f is reduced if $\xi(f) \in \mathcal{F}_{\mathbb{Z}(i)}$.

Proposition 2. *Let $f(x,y) = ax\bar{x} + b\bar{x}y + \bar{b}x\bar{y} + cy\bar{y}$ be a binary quadratic Hermitian form. Then f is reduced over F if and only if*

$$-\frac{a}{2} \leq \operatorname{Re}(b) \leq \frac{a}{2}, \qquad 0 \leq \operatorname{Im}(b) \leq \frac{a}{2}, \qquad a \leq c.$$

Moreover, $||b|| \leq a \leq c$.

Proof. The binary quadratic Hermitian form f is reduced if $\xi(f) \in \mathcal{F}_{\mathbb{Z}(i)}$, i.e.,

$$\xi(f) = -\frac{b}{a} + \frac{\sqrt{\Delta}}{a} \cdot j \in \mathcal{F}_{\mathbb{Z}(i)}.$$

Denote by $z = -\frac{b}{a}$ and $t = \frac{\sqrt{\Delta}}{a}$. By the description of fundamental domain $\mathcal{F}_{\mathbb{Z}(i)}$ given in Eq. (4) we have $-\frac{a}{2} \leq \operatorname{Re}(b) \leq \frac{a}{2}$, $0 \leq \operatorname{Im}(b) \leq \frac{a}{2}$, and $||z||^2 + t^2 \geq 1$. Since $||z||^2 + t^2 \geq 1$ we have

$$1 \leq \frac{||b||^2}{a^2} + \frac{\Delta}{a^2} = \frac{||b||^2 + ac - ||b||^2}{a^2} = \frac{c}{a}$$

i.e. $a \leq c$. Now consider

$$||b||^2 = \operatorname{Re}(b)^2 + \operatorname{Im}(b)^2 \leq \frac{a^2}{4} + \frac{a^2}{4} = \frac{a^2}{2}.$$

Hence, $||b|| \leq \frac{a\sqrt{2}}{2} \leq a \leq c$, which proves the last part. □

By discreteness of $\mathbb{Z}[i]$, the elements a and b may take on only finitely many values. The discriminant $\Delta = ac - b\bar{b}$, hence c is determined by a and b. Therefore, c may take on only finitely many values too.

In [5, Table 2] we list (count) the number of reduced binary quadratic Hermitian forms with fixed discriminant. To each tuple $[a, b, c]$ corresponds a binary quadratic Hermitian form

$$Q(X, Z) = aX\bar{X} + \bar{b}X\bar{Z} + b\bar{X}Z + cZ\bar{Z}.$$

In the first column is given the discriminant, in the second one the reduced forms $[a, b, c]$ with that given discriminant, and in the third column the number of reduced forms.

Proposition 3. *Let $f \in \operatorname{Her}^+(\mathcal{O}_F)$. If f is reduced over F then f has minimal height in its Γ_F-orbit.*

Proof. Since f is defined over the Gaussian integers, as shown in [5, Example 1], the height is just $\operatorname{H}(f) = \max\{ |x_j|_\infty \}$. Since f is reduced, from above Prop. Prop. 2 we have that

$$\operatorname{H}_F(f) = \max\{||b||, |a|, |c|\} = c.$$

We need to show that this is the minimal height on its Γ_F-orbit. In analogy with the case of binary quadratic forms defined over the reals we will prove it only for the generators of Γ_F. Let Q be the matrix associated to the given binary quadratic Hermitian form. Consider first the action of S on f. We have

$$Q^S = S^\star Q S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b \\ a+\bar{b} & a + 2\operatorname{Re}(b) + c \end{pmatrix}$$

and

$$\operatorname{H}_F(f^S) = \max\left\{ ||a||, ||a+b||, ||a+\bar{b}||, ||a + 2\operatorname{Re}(b) + c|| \right\}.$$

Since $a > 0$, and $-\frac{a}{2} \leq \operatorname{Re}(b) \leq \frac{a}{2}$, then $||a + 2\operatorname{Re}(b) + c|| \geq c$. Therefore, $\mathrm{H}_F(f^S) \geq \mathrm{H}_F(f)$.

Let T act on f. The associated matrix to f^T is as follows

$$Q^T = T^\star Q T = \begin{pmatrix} -i & 0 \\ 1 & i \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} a & -ai + b \\ b + ic & a + 2\operatorname{Im}(b) + c \end{pmatrix}$$

and

$$\mathrm{H}_F(f^T) = \max \left\{ ||a||, ||-ai + b||, ||b + ic||, ||a + 2\operatorname{Im}(b) + c|| \right\}.$$

But $a > 0$, and $0 \leq \operatorname{Im}(b) \leq \frac{a}{2}$, hence $||a + 2\operatorname{Im}(b) + c|| \geq c$. Therefore, $\mathrm{H}_F(f^T) \geq \mathrm{H}(f)$.

Let U act on f. The associated matrix to the form f^U is

$$Q^U = U^\star Q U = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} a & -b \\ -\bar{b} & c \end{pmatrix}.$$

Hence, $\mathrm{H}_F(f^U) = \mathrm{H}_F(f)$.

Lastly, let W act on f. The matrix associated to the new form f^W is

$$Q^W = W^\star Q W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c & -\bar{b} \\ -b & a \end{pmatrix},$$

and the height of the new form does not change. Hence, we conclude $\mathrm{H}_F(f^M) \geq \mathrm{H}_F(f)$ for any $M \in \Gamma_F$. Therefore, f has minimal height in its Γ_F-orbit. $\square$

In [5, Table 2] we display a table of classes of binary quadratic Hermitian forms with given discriminant. We list reduced forms representative of classes given by $[a, b, c]$ with a given discriminant Δ. The algorithm to compute this forms is similar with the one for computing binary quadratic forms with a given discriminant.

5. Reduction of higher degree binary forms

Next we give an algorithm such that for any form f with degree $n > 2$ defined over a ring of integers $\mathcal{O}_K$, we find a form with minimal height $\mathrm{H}(f)$ in its $\Gamma_{\mathcal{O}_K}$-orbit.

5.1. **Julia quadratic of binary forms.**

Julia quadratic was introduced in 1917 by Gaston Julia in his PhD thesis; see [23]. It did not get the attention that it deserved. Indeed Julia became known for most of his other work on Julia sets and fractals. However, in 1999 Cremona [14] used ideas of Julia to explore the reduction for cubic binary forms. More recently Cremona and Stoll in [34] gave a generalization of Julia's work for binary forms defined over $\mathbb{C}$.

Julia quadratic of binary forms with real coefficients. We will motivate and define the Julia quadratic of a binary form of degree $n \geq 2$ with real coefficients. We will try to follow as closely as possible the approach and notation used in Julia's original paper [23].

Let $f(x, y) \in \mathbb{R}[x, y]$ be a degree n binary form given as follows:

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$$

and suppose that $a_0 \neq 0$. Let the real roots of $f(x, y)$ be α_i, for $1 \leq i \leq r$ and the pair of complex roots β_j, $\bar{\beta}_j$ for $1 \leq j \leq s$, where $r + 2s = n$. The form can be factored as

$$f(x,1)=\prod_{i=1}^{r}(x-\alpha_i)\cdot\prod_{i=1}^{s}(x-\beta_i)(x-\bar{\beta}_i). \tag{13}$$

The ordered pair (r,s) of numbers r and s is called the **signature** of the form f.

We associate to f the two quadratics $T_r(x,1)$ and $S_s(x,1)$ of degree r and s respectively given by the formulas

$$T_r(x,1)=\sum_{i=1}^{r}t_i^2(x-\alpha_i)^2, \quad \textit{and} \quad S_s(x,1)=\sum_{j=1}^{s}2u_j^2(x-\beta_j)(x-\bar{\beta}_j), \tag{14}$$

where t_i, u_j are to be determined. Then

$$\begin{aligned} T_r(x,1)&=\left(\sum_{i=1}^{r}t_i^2\right)x^2-2\left(\sum_{i-1}^{r}t_i^2\alpha_i\right)x+\left(\sum_{i-1}^{r}t_i^2\alpha_i^2\right)\\ S_s(x,1)&=2\left(\sum_{j=1}^{s}u_j^2\right)x^2-4\left(\sum_{j=1}^{s}u_j^2\operatorname{Re}(\beta_j)\right)x+2\left(\sum_{j=1}^{s}u_j^2\cdot||\beta_j||^2\right). \end{aligned} \tag{15}$$

For a binary form f of signature (r,s) the quadratic Q_f is defined as

$$Q_f(x,1)=T_r(x,1)+S_s(x,1). \tag{16}$$

Let $\beta_i=a_i+b_i\cdot I$, for $i=1,\dots,s$. Then Q_f can be written as

$$\begin{aligned} Q_f&=\sum_{i=1}^{r}t_i^2(x^2-2\alpha_i\,x+\alpha_i^2)+2\sum_{j=1}^{s}u_j^2\,\left(x^2-2a_j\cdot x+(a_j^2+b_j^2)\right),\\ &=\left(\sum_{i=1}^{r}t_i^2+2\sum_{j=1}^{s}u_j^2\right)x^2-2\left(\sum_{i=1}^{r}\alpha_it_i^2+2\sum_{j=1}^{s}a_ju_j^2\right)x\\ &+\left(\sum_{i=1}^{r}t_i^2\alpha_i^2+2\sum_{j=1}^{s}u_j^2\cdot(a_j^2+b_j^2)\right). \end{aligned} \tag{17}$$

The discriminant of Q_f is a degree 4 homogenous polynomial in $t_1,\dots t_r$, $u_1,\dots,u_s$. We would like to pick values for $t_1,\dots t_r,u_1,\dots,u_s$ such that this discriminant is square free and minimal. Then we can use the reduction theory of quadratics (with square free, minimal discriminant) to determine the reduced form for Q_f.

For quadratics T and S in Eq. (14) we define

$$\theta_T=\frac{a_0^2\cdot\Delta_T}{t_1^2\cdots t_r^2}, \qquad \theta_S=\frac{a_0^2\cdot\Delta_S}{u_1^4\cdots u_s^4} \tag{18}$$

Notice that T_r and S_s are given recursively as

$$T_r=T_{r-1}+t_r^2(x-\alpha_r)^2, \qquad S_s=S_{s-1}+u_s^4\left(x^2-2a_sx+(a_s^2+b_s^2)\right)$$

The next lemma gives formulas computing the discriminants of T and S.

Lemma 2. *Let T_r and S_s be quadratics given by*

$$T_r(x,1)=\sum_{i=1}^{r}t_i^2(x-\alpha_i)^2, \quad \textit{and} \quad S_s(x,1)=\sum_{j=1}^{s}2u_j^2(x-\beta_j)(x-\bar{\beta}_j), \tag{19}$$

where $\beta_i=a_i+I\cdot b_i$, for $i=1,\dots,s$. Then $T_r\in V_{2,\mathbb{R}}^+$ and $S_s\in V_{2,\mathbb{R}}^+$.

Moreover,

$$
\begin{aligned}
\Delta(T_r) &= -4\,(t_1^2 \cdots t_r^2) \sum_{\substack{i,j=1\\ i\neq j\\ n_l\neq i, n_l\neq j}}^{r} \frac{(\alpha_i-\alpha_j)^2}{t_{n_1}^2\cdots t_{n_l}^2\cdots t_{n_{r-2}}^2} = -4\sum_{i<j}^{r} t_i^2 t_j^2\,(\alpha_i-\alpha_j)^2, \\
\Delta(S_s) &= -16\left(\sum_{i<j} u_i^2 u_j^2\left[(a_i-a_j)^2+(b_i^2+b_j^2)\right] + \sum_{j=1}^{s} u_j^4 b_j^2\right).
\end{aligned}
\tag{20}
$$

Let $f \in V_{n,\mathbb{R}}$ with signature (r,s) and equation as in Eq. (13). Then Q_f is a positive definite quadratic form with discriminant $\mathfrak{D}_f$ given by the formula

$$
\mathfrak{D}_f = \Delta(T_r)+\Delta(S_s) - 8\sum_{i,j} t_i^2 u_j^2\left((\alpha_i-a_j)^2+b_j^2\right). \tag{21}
$$

From the above formula it can be seen that $\mathfrak{D}_f$ is expressed in terms of the root differences. Hence, $\mathfrak{D}_f$ is fixed by all the transpositions of the roots. However, it is not an invariant of the binary form. In order to get an invariant we need to fix it by all symmetries of the roots, hence by an element of order n. It will be seen later that $\mathfrak{D}_f^n$ is an invariant of the binary form f.

The above remark motivates the following definition. We define the θ_0 of a binary form as follows

$$
\theta_0(f) = \frac{a_0^2 \cdot |\mathfrak{D}_f|^{n/2}}{\prod_{i=1}^r t_i^2 \,\prod_{j=1}^s u_j^4}.
$$

Next we continue with the general theory. Consider the function

$$
\theta_0(t_1,\dots,t_r,u_1,\dots,u_s)
$$

as a multivariable function in the variables $t_1,\dots,t_r,u_1,\dots,u_s$. We would like to pick these variables such that Q_f is a reduced quadratic, hence $\mathfrak{D}_f$ is minimal. This is equivalent to $\theta_0(t_1,\dots,t_r,u_1,\dots,u_s)$ obtaining a minimal value.

Proposition 4. *The function $\theta_0 : \mathbb{R}^{r+s} \to \mathbb{R}$ obtains a minimum at a unique point $(\bar{t}_1,\dots,\bar{t}_r,\bar{u}_1,\dots,\bar{u}_s)$.*

Proof. Julia in his thesis [23] proves existence and Stoll, and Cremona prove uniqueness in [34]. ☐

Choosing $(\bar{t}_1,\dots,\bar{t}_r,\bar{u}_1,\dots,\bar{u}_s)$ that make θ_0 minimal gives a unique positive definite quadratic $Q_f(X,Z)$. We call this unique quadratic $Q_f(X,Z)$ for such a choice of $(\bar{t}_1,\dots,\bar{t}_r,\bar{u}_1,\dots,\bar{u}_s)$ the **Julia quadratic** of $f(X,Z)$, denote it by $\mathcal{J}_f(X,Z)$, and the quantity $\theta_f := \theta_0(\bar{t}_1,\dots,\bar{t}_r,\bar{u}_1,\dots,\bar{u}_s)$ the **Julia invariant**. From the previous remarks, this is well defined.

The following lemma shows that θ is an invariant of binary forms and $\mathcal{J}$ a covariant of order 2.

Lemma 3. *Consider $\mathrm{GL}_2(\mathbb{C})$ acting on $V_{n,\mathbb{R}}$. Then θ is an invariant of binary forms and $\mathcal{J}$ is a covariant of order 2.*

We will prove this lemma in the next section for the general case, i.e. for binary forms over $\mathbb{C}$. Next, we make the necessary adjustments such that the above construction will work for binary forms with complex coefficients as well.

Julia's quadratic for binary forms with complex coefficients. Suppose we are given a binary form $f \in V_{n,\mathbb{C}}$ with $f(x,y) = \sum_{i=0}^{n} x^{n-i} y^i$ and assume that $a_0 \neq 0$. Then $f(x,y)$ can be factored as

$$f(x,y) = a_0 (y_1 x - x_1 y)(y_2 x - x_2 y) \cdots (y_n x - x_n y), \tag{22}$$

for $[x_i, y_i] \in \mathbb{P}^1$, $i = 1, \dots n$. Construct a quadratic form

(23)

$$\begin{aligned} Q(x,y) &= \sum_{i=1}^{n} t_i^2 \cdot ||y_i x - x_i y||^2 \\ &= \left(\sum_{i=1}^{n} t_i^2 ||y_i||^2 \right) x\bar{x} - \left(\sum_{i=1}^{n} t_i^2 y_i \bar{x}_i \right) x\bar{y} - \left(\sum_{i=1}^{n} t_i^2 x \bar{y}_i \right) \bar{x} y + \left(\sum_{i=1}^{n} t_i^2 \cdot ||x_i||^2 \right) y\bar{y} \end{aligned}$$

where t_j are non-zero real numbers that have to be determined. Computing the discriminant of the quadratic $Q(X,Z)$ and simplifying it we get

$$\mathfrak{D}_f = \sum_{1=i<j=n} t_i^2 t_j^2 \cdot ||y_i x_j - x_i y_j||^2 = \sum_{1=i<j=n} t_i^2 t_j^2 \cdot ||\beta_{ij}||^2. \tag{24}$$

Note that $||\beta_{ij}|| := ||y_i x_j - x_i y_j||$. Since the leading coefficient of Q and $\mathfrak{D}_f$ are both positive then Q is a positive definite quadratic Hermitian form. We define the quantity θ_0 as

$$\theta_0(Q_f) = \frac{||a_0||^2 \cdot \mathfrak{D}_f^{n/2}}{t_1^2 \cdots t_n^2}.$$

Consider θ_0 as a function

$$\begin{aligned} \theta_0 : \mathbb{P}^{n-1} \backslash \{(0, \dots, 0)\} &\to \mathbb{P}^1 \\ (t_1, \dots, t_n) &\mapsto \theta_0(t_1, \dots, t_n). \end{aligned}$$

Since this is a function defined on $\mathbb{P}^{n-1}$ then we take its domain to be

$$D = \left\{ (t_1, \dots, t_n) \in \mathbb{P}^n : \; t_1^2 \cdot t_2^2 \cdots t_n^2 = 1 \right\}.$$

We would like to choose $t_1, \dots, t_n$ such that Q_f is a reduced quadratic, hence a quadratic with minimal discriminant. Then θ_0 obtains a minimum exactly when $\mathfrak{D}_f$ obtains a minimum, under the assumption $t_1^2 \cdots t_n^2 = 1$. Our next task is to determine in what values for $(t_1, \dots, t_n)$ this minimum occurs. For simplicity denote by $h = \mathfrak{D}_f$. To find the critical points in the interior of D we need to solve $\nabla_h = 0$, i.e.

$$2t_i \sum_{\substack{j=1 \\ j \neq i}} t_j^2 \cdot ||y_i x_j - x_i y_j||^2 = 0, \qquad i = 1, \dots n.$$

Note that the only critical point in the interior D° is the tuple $(0, \dots, 0)$, which is not in the domain.

Next, determine the critical points on the boundary of D. Denote by $g = \prod_{i=1}^{n} t_i^2 = 1$. Using Lagrange multipliers we have to solve the system

$$\begin{cases} \nabla_h = \lambda \nabla_g \\ t_1^2 \cdots t_n^2 = 1 \end{cases}$$

for $\lambda \neq 0$. For convenience denote

$$\boxed{u_i = t_i^2 \quad \textit{and} \quad \alpha_{i,j} = ||\beta_{i,j}||^2 = ||y_i x_j - x_i y_j||^2}$$

and we have

$$\begin{cases} \sum_{\substack{j=1\\ i\neq j}}^n u_j \cdot \alpha_{i,j} = \lambda \cdot \prod_{i\neq j} u_j, & i=1,\dots,n \\ \prod_{i=1}^n u_i = 1 \end{cases}$$

or equivalently

$$\begin{cases} u_i \sum_{\substack{j=1\\ i\neq j}}^n u_j \cdot \alpha_{i,j} = \lambda \\ \prod_{i=1}^n u_i = 1 \end{cases} \tag{25}$$

Summing up the first n-equations of the system in Eq. (25), we get

$$\sum_{\substack{i,j=1\\ i<j}}^n u_i\, u_j\, \alpha_{i,j} = n\cdot\lambda \tag{26}$$

Then the left hand side of Eq. (26) is equal to $2\cdot\mathfrak{D}_f$. Therefore, $2\cdot\mathfrak{D}_f = n\cdot\lambda$ and $\lambda = \frac{2\cdot\mathfrak{D}_f}{n}$

$$\lambda = \frac{2}{n}\cdot\sum_{i<j} u_i u_j \alpha_{i,j}. \tag{27}$$

Substituting λ in the system in Eq. (25) we have

$$\begin{cases} n\cdot u_1\left(u_2\alpha_{1,2}+u_3\alpha_{1,3}+\cdots+u_n\alpha_{1,n}\right) = 2\cdot\sum_{i<j} u_i u_j \alpha_{i,j} \\ n\cdot u_2\left(u_1\alpha_{1,2}+u_3\alpha_{2,3}+\cdots+u_n\alpha_{2,n}\right) = 2\cdot\sum_{i<j} u_i u_j \alpha_{i,j} \\ \vdots \\ n\cdot u_n\left(u_1\alpha_{2,n}+u_3\alpha_{3,n}+\cdots+u_{n-1}\alpha_{n-1,n}\right) = 2\cdot\sum_{i<j} u_i u_j \alpha_{i,j} \\ u_1\cdot u_2\cdots u_n = 1. \end{cases} \tag{28}$$

Consider the first row. We have

$$\begin{aligned} u_1u_2\alpha_{1,2}+u_1u_3\alpha_{1,3}+\cdots+u_1u_n\alpha_{1,n} =& \frac{2}{n}\cdot\big(u_1u_2\alpha_{1,2}+\cdots+u_1u_n\alpha_{1,n}+ \\ & u_2u_3\alpha_{2,3}+\cdots+u_2u_n\alpha_{2,n}+ \\ & \vdots \\ & + u_{n-1}u_n\alpha_{n-1,n}\big) \end{aligned}$$

and combining like terms we have

$$\begin{aligned} (n-2)(u_1u_2\alpha_{1,2}+u_1u_3\alpha_{1,3}+\cdots+u_1u_n\alpha_{1,n}) =& 2\cdot\big(u_2u_3\alpha_{2,3}+\cdots+u_2u_n\alpha_{2,n}+ \\ & u_3u_4\alpha_{3,4}+\cdots+u_3u_n\alpha_{2,n}+ \\ & \vdots \\ & + u_{n-1}u_n\alpha_{n-1,n}\big). \end{aligned}$$

The i'th row for $i=1,\dots,n$ will look like

$$(n-2)\cdot\sum_{i<j} u_i u_j \alpha_{i,j} = 2\cdot\sum_{\substack{l<k\\ l,k\neq i}} u_l u_k \alpha_{l,k}. \tag{29}$$

Remark 1. *We can make the substitution $\gamma_{i,j} = u_i u_j \alpha_{i,j}$, since in the formula for the Julia invariant these are the terms that appear. Then the system becomes a linear system with n equations and $\binom{n}{2}$ variables. Obviously $n = \binom{n}{2}$, when $n = 3$. Hence, the case of cubics is very easy.*

Let V be the variety defined by the Eq. (28). We have the following result.

Theorem 7. *V is a zero dimensional variety over $\mathbb{C}$. Moreover, V has exactly one real point given by*

$$u_i = \frac{2}{n} \cdot \frac{t^2}{(||z-\alpha_i||^2 + t^2)},$$

where t and z satisfy the following system

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \frac{t^2}{||z-\alpha_j||^2 + t^2} = \frac{n}{2} \\ \displaystyle\sum_{j=1}^{n} \frac{z-\alpha_j}{||z-\alpha_j||^2 + t^2} = 0 \end{cases} \tag{30}$$

Proof. A solution to the Eq. (28) determines the Julia quadratic and therefore a point in $\mathcal{H}_3$. Let $(z,t) \in \mathcal{H}_3$ be such a point. The quadratic associated to (z,t) is equal to the Julia quadratic as in Eq. (23). Hence,

$$Q(x,y) = S\left(|x+zy|^2 + t|y|^2\right)$$

where

$$S = \sum_{i=1}^{n} t_i^2, \quad Sz = \sum_{i=1}^{n} \alpha_i t_i^2, \quad S(||z||^2 + t^2) = \sum_{i=1}^{n} ||\alpha_i||^2 t_i^2$$

and $\frac{1}{4}\mathfrak{D}_f = S^2 t^2$. Consider Eq. (25). Note that

$$\sum_{j=1}^{n} u_j \alpha_{i,j} = S(||z-\alpha_i||^2 + t^2)$$

because

$$\begin{aligned} \sum_{j=1}^{n} u_j \alpha_{i,j} &= \sum_{j=1}^{n} u_j ||\alpha_i - \alpha_j||^2 = \sum_{j=1}^{n} u_j \left(||\alpha_i||^2 - \alpha_i \bar{\alpha}_j - \bar{\alpha}_i \alpha_j + ||\alpha_j||^2\right) \\ &= ||\alpha_i||^2 \sum_{j=1}^{n} u_j - \alpha_i \sum_{j=1}^{n} u_j \bar{\alpha}_j - \bar{\alpha}_i \sum_{j=1}^{n} u_j \alpha_j + \sum_{j=1}^{n} u_j ||\alpha_j||^2 \\ &= ||\alpha_i||^2 \cdot S - \alpha_i \cdot \bar{z} S - \bar{\alpha}_i \cdot z S + S(||z||^2 + t^2) \\ &= S\left(||\alpha_i||^2 - \alpha_i \cdot \bar{z} - \bar{\alpha}_i \cdot z + ||z||^2\right) + St^2 = S\left(||z-\alpha_i||^2 + t^2\right). \end{aligned}$$

Hence, the system in Eq. (25) becomes

$$\begin{cases} u_1 \cdot S(||z-\alpha_1||^2 + t^2) = \lambda \\ u_2 \cdot S(||z-\alpha_2||^2 + t^2) = \lambda \\ \vdots \\ u_n \cdot S(||z-\alpha_n||^2 + t^2) = \lambda \\ u_1 \cdot u_2 \cdots u_n = 1. \end{cases} \tag{31}$$

Since $2\,\mathfrak{D}_f = n\lambda$, we have $\lambda = \frac{2S^2t^2}{n}$ and

$$u_i = \frac{2}{n} \cdot \frac{S^2t^2}{S(||z-\alpha_i||^2+t^2)},$$

for each $i = 1, \ldots, n$. We can assume that $S = 1$ and take the Julia quadratic to be a monic. Then

$$u_i = \frac{2}{n} \cdot \frac{t^2}{(||z-\alpha_i||^2+t^2)}.$$

Since $S = \sum_{i=1}^n t_i^2$ and $Sz = \sum_{i=1}^n \alpha_i t_i^2$, the system in Eq. (28) becomes as follows

$$\begin{cases} \displaystyle\sum_{j=1}^n \frac{t^2}{||z-\alpha_j||^2+t^2} = \frac{n}{2} \\ \displaystyle\sum_{j=1}^n \frac{z-\alpha_j}{||z-\alpha_j||^2+t^2} = 0. \end{cases} \tag{32}$$

To prove the theorem it is enough to show that this system has a unique solution (z, t) for $z \in \mathbb{C}$, and $t \in \mathbb{R}^+$. We make the convenient substitution $t^2 = \bar{t}$. We have two equations of degree $2n$ and $2n-1$ in z and of degree n and $n-1$ in t, as displayed below:

$$F_1(\bar{t}, u) = 0 \quad \text{and} \quad F_2(\bar{t}, u) = 0 \tag{33}$$

By Prop. Prop. 4 this has a unique positive real root which is t. $\square$

Let $(\bar{u}_1, \ldots, \bar{u}_n) \in \mathbb{R}^n$ be the unique real point of V. From now on by θ_f we will denote the function θ_0 evaluated at this unique point. The quadratic $Q(f)$ for the above values $(\bar{u}_1, \ldots, \bar{u}_n)$ will be denoted by $\mathcal{J}_f$ and is called **Julia's quadratic**.

Lemma 4. *Let $\mathrm{GL}_2(\mathbb{C})$ act on $V_{n,\mathbb{C}}$. Then the following are true:*
i) θ_f is an invariant
ii) $\mathfrak{D}_f^n$ is an invariant.

Proof. Let $f \in V_{n,\mathbb{C}}$ be a binary form which is factored over $\mathbb{C}$ as follows

$$f(x, y) = \prod_{i=1}^n (x - \alpha_i y).$$

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ act on f as follows

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

The roots of f^M are respectively $\gamma_i = M^{-1}\alpha_i$. Assume first that none of the roots of f go to infinity under M. Then the substitution for $(x - \alpha_i y)$ is

$$ax_1 + by_1 - \frac{a\gamma_i + b}{c\gamma_i + d} \cdot (cx_1 + dy_1) = (a - c\alpha_i)(x_1 - \gamma_i y_1).$$

Therefore,

$$f(ax_1 + by_1, cx_1 + dy_1) = A_0 \prod_{i=1}^n (x_1 - \gamma_i y_1)$$

where

$$A_0 = a_0 \prod_{i=1}^{n} (a - \alpha_i c). \tag{34}$$

Acting by the same matrix M on the positive definite binary quadratic Q_f associated to f we get

$$Q_f^M = \sum_{i=1}^{r} \tau_i^2 (x_1 - \gamma_i y_1)^2$$

where τ_i^2 is given as follows

$$\tau_i^2 = t_i^2 (a - \alpha_i c)^2.$$

Recall that $\mathfrak{D}_f := \Delta(Q_f)$, and when we act on a binary quadratic form by a matrix M, with $\det(M) = \lambda$, the determinant is fixed. Then

$$\theta_0(f^M) = \frac{A_0^2 \sqrt{\mathfrak{D}_{f^M}^n}}{\prod_{i=1}^n \tau_i^2} = \frac{\left[a_0 \prod_{i=1}^n (a - \alpha_i c)\right]^2 \cdot \sqrt{\mathfrak{D}_f^n}}{\prod_{i=1}^n t_i^2 (a - \alpha_i c)^2} = \frac{a_0^2 \cdot \sqrt{\mathfrak{D}_f^n}}{\prod_{i=1}^n t_i^2} = \theta_0(f).$$

Now, assume the first p real roots of $f(x,y)$ are equal to $\frac{a}{c}$, i.e. the first p-real roots of f go to infinity under M. Then the substitution for $(x - \alpha_i y)$ for $i = 1, \cdots, p$ becomes

$$ax_1 + by_1 - \frac{a}{c}(cx_1 + dy_1) = -\frac{y_1}{c}.$$

Hence,

$$F(ax_1 + by_1, cx_1 + dy_1) = A_0 \cdot y_1^p \prod_{i=1}^{n} (x_1 - \gamma_i y_1)$$

where

$$A_0 = \frac{(-1)^p}{c^p} \cdot a_0 \prod_{i=p+1}^{n} (a - \alpha_i c).$$

The positive definite binary quadratic form associated to $f(x_1, y_1)$ is

$$Q_f^M = \sum_{i=1}^{p} \tau_i^2 y_1^2 + \sum_{i=p+1}^{n} \tau_i^2 (x_1 - \gamma_i y_1)^2$$

where

$$\tau_i^2 = \begin{cases} \frac{t_i^2}{c^2} & i = 1, \cdots, p \\ t_i^2 (a - \alpha_i c)^2 & i = p+1, \cdots, n. \end{cases}$$

By calculating the Julia invariant of $f(x_1, y_1)$ and simplifying it we get

$$\begin{aligned} \theta_0(f^M) &= \frac{A_0^2 \sqrt{\mathfrak{D}_{f^M}^n}}{\prod_{i=1}^n \tau_i^2} \\ &= \frac{\left(\frac{(-1)^p}{c^p} \cdot a_0 \prod_{i=p+1}^n (a - \alpha_i c)\right)^2 \cdot \sqrt{\mathfrak{D}_f^n}}{\prod_{i=1}^p \frac{t_i^2}{c^2} \prod_{i=p+1}^n t_i^2 (a - \alpha_i c)^2} = \frac{a_0^2 \cdot \sqrt{\mathfrak{D}_f^n}}{\prod_{i=1}^n t_i^2} = \theta_0(f). \end{aligned}$$

Thus, $\theta_0(f^M) = \theta_0(f)$ and therefore θ_0 is an invariant. Part ii) is a direct consequence of the definition of θ. □

Corollary 3. *Let $f \in V_{n,\mathbb{C}}$ and F_f its field of moduli. Then,*

i) $\theta_f \in F_f$.

ii) $a_0^4 \, \mathfrak{D}_f^n \in F_f(\theta_f^2)$.

Proof. It is by definition that $\theta_f \in F_f$ and $\mathcal{J}_f$ has coefficients in $F_f[x, y]$. Part iii) is a consequence of the definition of θ_f. □

Problem 1. *An open question is to express θ in terms of generators of the rings of invariants for degree n binary forms or absolute invariants of f which determine the field of moduli of f.*

5.2. **Reducing binary forms of higher degree.** In this section we will describe reduction theory of higher degree binary forms. First, we will explain the case of binary forms with real coefficients and then its generalization to binary forms with complex coefficients.

Binary forms with real coefficients. To any form $f \in V_{n,\mathbb{R}}$ we associate a positive definite quadratic $\mathcal{J}_f \in V^+_{2,\mathbb{R}}$ as showed above. In Section 4 we proved that binary quadratic forms in $V^+_{2,\mathbb{R}}$ are in one-to-one correspondence with points in the upper half plane $\mathcal{H}_2$. Hence, we have the following maps

$$\begin{aligned} \zeta : V_{n,\mathbb{R}} &\to V^+_{2,\mathbb{R}} \to \mathcal{H}_2 \\ f &\mapsto \mathcal{J}_f \mapsto \xi(\mathcal{J}_f). \end{aligned}$$

We call this map the **zero map** and denote it by $\zeta(f) := \xi(\mathcal{J}_f)$. The map $\zeta : V_{n,\mathbb{R}} \to \mathcal{H}_2$ is $\mathrm{SL}_2(\mathbb{R})$-equivariant.

The proof of the above is easy and it will be proved in the next subsection for the more general case, i.e. binary forms with complex coefficients. A binary form $f \in V_{n,\mathbb{R}}$ is **reduced** if $\zeta(f) \in \mathcal{F}_2$. Next, we will adapt this to binary forms with complex coefficients.

Binary forms with complex coefficients. For any form $f \in V_{n,\mathbb{C}}$ the corresponding Julia quadratic is a positive definite Hermitian form. Previously we proved that binary quadratic forms in $\mathrm{Her}^+(\mathbb{C})$ are in a one-to-one correspondence with points in $\mathcal{H}_3$. Hence, we have the maps:

$$\begin{aligned} \zeta : V_{n,\mathbb{C}} &\longrightarrow \mathrm{Her}^+(\mathbb{C}) \longrightarrow \mathcal{H}_3 \\ f &\mapsto \mathcal{J}_f \mapsto \xi(\mathcal{J}_f) \end{aligned}$$

where ξ is as defined in Eq. (12). Note that $\xi(\mathcal{J}_f)$ is the point in $\mathcal{H}_3$ associated to the Hermitian form $\mathcal{J}_f$.

Lemma 5. *The map $j : V_{n,\mathbb{C}} \longrightarrow \mathrm{Her}^+(\mathbb{C})$ is an $\mathrm{SL}_2(\mathbb{C})$-equivariant map, i.e. for every $f \in V_{n,\mathbb{C}}$, $H \in \mathrm{Her}^+(\mathbb{C})$ and $M \in \mathrm{SL}_2(\mathbb{C})$ we have $j(f^M) = j(f)^M$ which is equivalent to saying $H_{f^M} = H_f^M$.*

Proof. We will prove it only for the generators of $\mathrm{SL}_2(\mathbb{C})$, i.e. for $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ where $m \in \mathbb{C}$. First, for $f \in V_{n,\mathbb{C}}$ such that

$$f = a_0(x - \alpha_1 y) \cdots (x - \alpha_n y)$$

and $H \in \mathrm{Her}^+(\mathbb{C})$ we want to prove that $H_{f^S} = H_f^S$. We have

$$f^S = A_0(x - \gamma_1 y) \cdots (x - \gamma_n y)$$

where $A_0 = a_0 \alpha_i^n$ and $\gamma_i = -\frac{1}{\alpha i}$.

The binary quadratic Hermitian form associated to f^S is

$$H_{f^S} = \sum \tau_i^2 ||x - \gamma_i y||^2.$$

On the other side,

$$H_f^S = \sum t_i^2 ||y - \alpha_i(-x)||^2 = \sum t_i^2 \left|\left|\alpha_i \left(x - \frac{y}{-\alpha_i}\right)\right|\right|^2$$
$$= \sum t_i^2 ||\alpha_i||^2 ||x - \gamma_i y||^2.$$

Notice that for $\tau_i^2 = t_i^2 ||\alpha_i||^2$, we have that $H_f^S = H_{f^S}$. Now let us show $H_{f^T} = H_f^T$. For $f = a_0(x - \alpha_1 y) \cdots (x - \alpha_n y)$ and T as above we have

$$f^T = A_0(x - \gamma_1 y) \cdots (x - \gamma_n y)$$

where $A_0 = a_0$ and $\gamma_i = \alpha_i - m$. The binary quadratic Hermitian form associated to f^T is

$$H_{f^T} = \sum \tau_i^2 ||x - \gamma_i y||^2.$$

On the other side,

$$H_f^T = \sum t_i^2 ||x + my - \alpha_i y||^2 = \sum t_i^2 ||x - (\alpha_i - m)y||^2 = \sum t_i^2 ||x - \gamma_i y||^2.$$

Hence, for $\tau_i^2 = t_i^2$ we have $H_f^T = H_{f^T}$ and we are done. $\square$

The map $\zeta : V_{n,\mathbb{C}} \to \mathcal{H}_3$ is $\mathrm{SL}_2(\mathbb{C})$-equivariant.

Let K be a field of definition of f. Without loss of generality assume that f has an integral model over $\mathcal{O}_K$. We call $f(x, y)$ to be **reduced** over K if $\zeta(f)$ is in a fixed fundamental domain for the action of Γ_K on $\mathcal{H}_3$, when such a fundamental domain exists.

Definition 4. *Let $f \in V_{n,\mathbb{C}}$ be such that it has an integral model over some algebraic number field K. We say $f(x, y)$ is reduced if $\zeta(f)$ is in a fixed fundamental domain for the action of $\mathrm{SL}_2(\mathcal{O}_K)$ on $\mathcal{H}_3$, when such a domain exists.*

Let f be a given degree n binary form. To find the reduced form in its $\mathrm{SL}_2(\mathcal{O}_K)$-orbit we compute $\zeta(f)$. If $\zeta(f)$ is in the fundamental domain $\mathcal{F}_{\mathcal{O}_K}$ we are done, the given form is the reduced one. Otherwise, compute $M \in \Gamma_{\mathcal{O}_K}$ such that $\zeta(f)^M \in \mathcal{F}_{\mathcal{O}_K}$ and $f^{M^{-1}}$ is the reduced form in its $\mathrm{SL}_2(\mathcal{O}_K)$-orbit.

A natural question to ask is the following; Does the reduced binary form computed this way have minimal height in its $\mathrm{SL}_2(\mathcal{O}_K)$-orbit? We will address this question in the remainder of this section.

Consider f a degree n binary form and K its minimal field of definition. Let $M \in \mathrm{SL}_2(\mathcal{O}_K)$ be a matrix such that f^M is reduced, i.e. $\bar{\xi}(f^M) \in \mathcal{F}_K$ where $\mathcal{F}_K$ is the fundamental domain of $\mathrm{SL}_2(\mathcal{O}_K)$ acting on $\mathcal{H}_3$.

First we give a bound on the height of the reduced binary form with respect to the Julia invariant.

Lemma 6. *Let f be a binary form with signature $(n, 0)$ factored as follows*

$$f(x, 1) = a_0 \prod_{i=1}^{n} (x - \alpha_i)$$

Then the height of this form can be bounded by Julia's invariant as

$$\mathrm{H}(f) \leq c \cdot \theta_f^{n/2}$$

where

$$c = \left(\frac{1}{3}\right)^{\frac{n^2}{4}} \left(\frac{4}{n-1}\right)^{\frac{n(n-1)}{2}} \frac{1}{a_0^n}$$

Proof. Let f be the reduced form given as above. It is easy to prove that the roots of the binary form can be bounded by the Julia invariant θ as follows

$$||\alpha_i||^2 \leq \frac{4^{n-1}}{(n-1)^{n-1}\, 3^{n/2}} \cdot \frac{1}{a_0^2} \cdot \theta_f,$$

see [23] for more details how to get this bound. Then the symmetric polynomials can be bounded as follows

$$\begin{aligned} s_r = \sum_{i=1}^{\binom{n}{r}} \alpha_i \cdots \alpha_r &\leq \binom{n}{r} \left(\sqrt{\frac{4^{n-1}}{(n-1)^{n-1}\, 3^{n/2}} \cdot \frac{1}{a_0^2} \cdot \theta_f} \right)^r \\ &\leq \binom{n}{r} \cdot \frac{1}{a_0^r} \cdot \sqrt{\frac{4^{r(n-1)}}{(n-1)^{r(n-1)}\, 3^{rn/2}}} \cdot \theta_f^{n/2} \end{aligned}$$

Hence, since the symmetric polynomials represent the coefficient of the binary form we have that

$$\mathrm{H}(f) \leq \binom{n}{r} \left(\frac{1}{3}\right)^{\frac{rn}{4}} \left(\frac{4}{n-1}\right)^{\frac{r(n-1)}{2}} \frac{1}{a_0^r} \cdot \theta_f^{n/2}$$

for all $r = 1, \dots, n$. Hence, $\mathrm{H}(f) \leq c \cdot \theta_f^{n//2}$ for all $r = 1, \dots, n$ and θ_f is minimal. Consider the function

$$f(n,r) = \binom{n}{r} \left(\frac{1}{3}\right)^{\frac{rn}{4}} \left(\frac{4}{n-1}\right)^{\frac{r(n-1)}{2}} \frac{1}{a_0^r}$$

We want to check if this is a decreasing or increasing function with respect to n

$$\begin{aligned} \frac{f(n+1,r)}{f(n,r)} &= \frac{\binom{n+1}{r} \left(\frac{1}{3}\right)^{\frac{(n+1)r}{4}} \left(\frac{4}{n-1}\right)^{\frac{rn}{2}} \frac{1}{a_0^r}}{\binom{n}{r} \left(\frac{1}{3}\right)^{\frac{rn}{4}} \left(\frac{4}{n-1}\right)^{\frac{r(n-1)}{2}} \frac{1}{a_0^r}} = \frac{\binom{n+1}{r}}{\binom{n}{r}} \left(\frac{1}{3}\right)^{\frac{r}{4}} \left(\frac{4}{n-1}\right)^{\frac{r}{2}} \\ &= \frac{n+1}{n+r-1} \left(\frac{1}{3}\right)^{\frac{r}{4}} \left(\frac{4}{n-1}\right)^{\frac{r}{2}} = \frac{n+1}{n+r-1}\, 2^r \left(\frac{1}{n-1}\right)^{\frac{r}{2}} \left(\frac{1}{3}\right)^{\frac{r}{4}} \end{aligned}$$

Since $n \geq 3$ and $r = 1, \dots, n$ we have that $\frac{f(n+1,r)}{f(n,r)} > 1$. Hence, $f(n,r)$ is an increasing function and the above bound becomes

$$\mathrm{H}(f) \leq \left(\frac{1}{3}\right)^{\frac{n^2}{4}} \left(\frac{4}{n-1}\right)^{\frac{n(n-1)}{2}} \frac{1}{a_0^n} \cdot \theta_f^{n/2}.$$

This completes the proof. □

In the following remark we will see that for binary cubics it is possible to express this bound in terms of the discriminant of the cubic and then we compare this bound with bounds obtained in [18].

Remark 2. *If we consider a binary cubic with signature* $(3,0)$ *then from Lem. 6 we have*

$$\mathrm{H}(f) \leq 2^3 \left(\frac{1}{3}\right)^{\frac{9}{4}} \frac{1}{a_0^3} \cdot \theta_f^{3/2}$$

Moreover, $\theta_f = a_0^6\, 3^{\frac{3}{2}}\, |\Delta_f|^{\frac{1}{2}}$*, (cf. Lem. 10). We can express the above bound in terms of the discriminant of the binary form* f

$$\mathrm{H}(f) \leq 2^3\, a_0^6 \;\cdot |\Delta_f|^{3/4}.$$

In [18, Thm 2, pg 162] *it is proved that for a binary form* f

$$\mathrm{H}(f) \leq C \cdot |\Delta_f|^{\frac{21}{2}},$$

where C *is some constant.*

The results in [18] are in line with previous results of the author and his collaborators in bounding the height of the binary forms in terms of the discriminants. There are many results comparing the height $\mathrm{H}(f)$ and Δ_f by many authors, including Mordell [28], Lewis [26], Mahler [27], Györy [20], Birch [9], Bombieri [10], and others.

5.3. **The minimal absolute height of binary forms.** Let K be a number field and $\mathcal{O}_K$ its ring of integers. We want to develop a reduction theory in the following sense: given a binary form $f(x, y)$ over $\mathcal{O}_K$ we determine its integral model with minimal height $\mathrm{H}(f)$ over $\overline{K}$.

Lemma 7. *Let* f *and* g *be two binary forms of degree* n *and* M *a matrix in* $\mathrm{SL}_2(\mathcal{O}_K)$ *such that* $g = f^M$*. Associate to these binary forms* f *and* g *respectively the Julia quadratics* $\mathcal{J}_f$ *and* $\mathcal{J}_g$*. Then the following holds:*

i) $\mathcal{J}_g = \mathcal{J}_f^M$

ii) $\Delta_{\mathcal{J}_f} = \Delta_{\mathcal{J}_g}$

Proof. The proof is trivial. Part i) follows directly from Lem. 5 and part ii) is true since we are acting with a matrix of discriminant one. □

Hence, the discriminant $\mathfrak{D}_f$ of the Julia quadratic is an invariant of the binary form. An interesting problem to consider would be to express $\mathfrak{D}_f$ in terms of the generators of the invariant ring $\mathcal{R}_n$.

The following theorem gives us a method to find the form with minimal height among all $\mathrm{SL}_2(\mathcal{O}_K)$-orbits.

Theorem 8. *Let* f *be a degree* n *binary form defined over* K *and* $\mathcal{J}_f$ *its Julia quadratic,* $\mathfrak{D}_f$ *its discriminant, and* $L = K(\mathfrak{D}_f)$*. Then* $[L : K] \leq n$*. Let* r *be the class number of* $\mathcal{J}_f$ *over* L *and* $M_1, \dots, M_r$ *the matrices with entries in* $\mathrm{SL}_2(\mathcal{O}_K)$ *that send* $\mathcal{J}_f$ *respectively to* $\{J_1, \dots, J_r\}$*. The form* f^{M_j} *for some* $j = 1, \dots, r$ *has minimal height over* $\mathrm{SL}_2(\mathcal{O}_K)$*.*

Proof. Let $\mathfrak{D}_f = \Delta_{\mathcal{J}_f}$ be the discriminant of the Julia quadratic associated to the degree n binary form. From Cor. Cor. 2 for any $\Delta \in \mathcal{O}_L$ with $\Delta \neq 0$ the set $V_{2,\mathcal{O}_L}(\Delta)$, i.e. the set of binary quadratics with that fixed discriminant, splits into finitely many $\mathrm{SL}_2(\mathcal{O}_L)$-orbits. Assume r is the class number of $\mathcal{J}_f$ over L and $\{J_1, \dots, J_r\}$ are representative reduced quadratics of each of these orbits. Let

$$\{M_1, \dots, M_r\} \in \mathrm{SL}_2(\mathcal{O}_L) \;\text{ such that }\; \mathcal{J}_f^{M_i} = J_i.$$

Act with the same matrices on the form f to get $f^{M_1}, \dots, f^{M_r}$, these are well defined from Lem. 7. The form with minimal height over all $\mathrm{SL}_2(\mathcal{O}_L)$-orbits will be the one with smallest height among $\{f^{M_1}, \dots, f^{M_r}\}$. This way we are finding the "best" binary form amongst all $\mathrm{SL}_2(\mathcal{O}_L)$-orbits. □

Once we find the "best" binary form amongst all $\mathrm{SL}_2(\mathcal{O}_L)$-orbits we can lower the height of the reduced form if we consider diagonal matrices with entries in $\mathcal{O}_K$. This is done as follows. Let f be a reduced form of degree $n \geq 3$ given by

$$f = a_n x^n + \cdots + a_0 y^n,$$

where $a_0, \dots, a_n \in \mathcal{O}_K$. Consider $M = \text{diag }(\alpha, \beta)$ the diagonal matrix with $\alpha, \beta \in \mathcal{O}_K$. Hence, $f^M = (\alpha x, \beta y)$.

Consider $f(\alpha x, y)$. The height $\mathrm{H}(f)$ can be lowered only if all coefficients of $f(\alpha x, y)$ have a common factor. Hence, we must choose α such that $\alpha \,|\, a_0$.

By the same argument, we choose β such that $\beta \,|\, a_n$. Obviously there are only finitely many choices for $M = \text{diag }(\alpha, \beta)$. Among all such choices we choose M that gives the smallest height. Obviously, $M \notin \mathrm{SL}_2(\mathcal{O}_K)$ therefore acting with M on the reduced form will lower the height. Hence, we have the following:

Theorem 9. *Let $f = \sum_{i=0}^n a_i x^i y^{n-i}$ be a reduced binary form. Choose $M =$ diag (α, β) such that $\alpha \,|\, a_0$ and $\beta \,|\, a_n$ and*

$$\mathrm{H}(f^M) = \min \left\{ \mathrm{H}\left(f^{\,diag\ (\alpha,\beta)} \right) \right\}$$

Then $\mathrm{H}(f^M) < \mathrm{H}(f)$.

Proof. Let $f = \sum_{i=0}^n a_i x^i y^{n-i}$ be a reduced binary form. Pick α and β such that $\alpha \,|\, a_0$ and $\beta \,|\, a_n$. Then

$$f(\alpha x, \beta y) = \sum_{i=0}^n a_i \alpha^i \beta^{n-i}\, x^i y^{n-i}$$

The content of this new polynomial is $\gcd(a_0, a_1 \alpha \beta^{n-1}, \dots, a_n \alpha^n)$. We choose the form with the smallest height among all primitives of $f(\alpha x, \beta y)$, where α, β are as above. ☐

5.4. **An algorithm to find the minimum absolute height.** We put everything together in the following algorithm, which finds the form with minimal height among all $\mathrm{GL}_2(\mathcal{O}_K)$-orbits is as follows.

Algorithm: Computing the binary form with minimal absolute height.

Input: A degree n binary form $f(x, y) \in V_{n, \mathcal{O}_K}$
Output: A binary form $F \in V_{n, \mathcal{O}_K}$ which is $\mathrm{GL}_2(\bar{K})$-equivalent to f and has minimal absolute height.

Step 1: Compute the Julia quadratic $\mathcal{J}_f$ associated to the binary form f, as explained in Eq. (5.1).
Step 2: Compute the zero map $\xi(\mathcal{J}_f) \in \mathcal{H}$, using Eq. (12).
Step 3: Find the matrix A such that $\xi(\mathcal{J}_f)^{A^{-1}} \in \mathcal{F}_{\mathcal{O}_K}$.
Step 4: Assign $f := \textbf{red}\ (f) = f^A$ and $J := J_f^{A^{-1}}$.
Step 5: Compute the discriminant $\mathfrak{D}_f$ of the quadratic form J.
Step 6: Let $L := K(\mathfrak{D}_f)$ and $h_L(\mathcal{J}) := r$ be the class number of J over L.
Step 7: Determine all quadratics $\{J_1, \dots, J_r\}$ equivalent to J over L and let $M_1, \dots, M_r \in \mathrm{GL}_2(L)$ be the matrices such that $J = J_i^{M_i}$, for $i = 1, \dots, r$.
Step 8: Compute the set of forms

$$f_1 := f^{M_1}, \dots, f_r := f^{M_r}.$$

Step 9: For each $i = 1, \ldots, r$, repeat steps 1-4 to compute **red** (f_i).
Step 10: For each $j = 1, \ldots, r$ and $f_j = \sum_{i=0}^n a_i x^i y^{n-i}$ do the following:
Choose $M = \text{diag}\ (\alpha, \beta)$ such that $\alpha \,|\, a_0$ and $\beta \,|\, a_n$ and pick
$g_j := f^{\text{diag}\ (\alpha,\beta)}$ such that

$$\mathrm{H}(f^M) = \min \left\{ \mathrm{H}\left(f^{\text{diag}\ (\alpha,\beta)} \right) \right\}$$

is minimal.

Step 11: Pick the form $F \in V_{n,\mathcal{O}_K}$ with smallest height among $g_1, \ldots, g_r$.
Return F

Next we highlight a few remarks about the efficiency of the algorithm.

Remark 3. 1) *For practical purposes computing $\zeta(f)$ numerically is satisfactory since we can find $A \in \Gamma$ such that $\zeta(f)^A \in \mathcal{F}$. Hence, the algorithm can be made rather efficient.*
2) *The reduced form* ***red*** *(f) has smaller coefficients and expected minimal height in its Γ-orbit.*

6. Computational aspects of reduction theory

In this section we explore some of the computational aspects of computing the Julia invariant and performing the reduction algorithm for higher degree binary forms. In this first section we give a brief description of the geometric aspects of the zero map and show that $\xi(f)$ corresponds to the centroid of a convex polygon determined by the roots, when to every root α_i we assign the weight t_i from the definition of the Julia quadratic as in Section 5. In the following sections we study forms with signature $(r, 0)$ and $(0, s)$ and compute the Julia quadratic for forms with degree $n = 3, 4, 5, 6$.

6.1. **Geometric interpretation of the zero map.** One approach to find the unique point (z, t) in the upper half space that makes θ minimal is solving the system given in Eq. (32). There is also another approach to find this point, a geometric approach. This is equivalent to finding the centroid of a convex polyhedron and weighted vertices. Julia in his thesis [23] solved the optimization problem geometrically for the case of binary cubics and quartics. He explicitly found $(t_1^2, \ldots, t_n^2)$, hence (z, t), for all possible signatures of binary cubics and quartics.

In this section, we will describe briefly this geometric approach and show how it generalizes to solving the optimization problem for higher degree binary forms. We will consider separately the case of binary forms with real and complex coefficients.

Binary forms with real coefficients. Let $f(x, 1)$ be a degree n binary form with real coefficients and signature (r, s), i.e. $\alpha_1, \ldots, \alpha_r$ its real roots and $\beta_1, \bar{\beta}_1, \ldots, \beta_s, \bar{\beta}_s$ its complex. Associate to it the quadratic

$$Q(x, 1) = \sum_{i=1}^{r} t_i^2 \cdot (x - \alpha_i)^2 + \sum_{i=1}^{s} 2u_i^2 \cdot (x - \beta_i)(x - \bar{\beta}_i)$$

where the t_i's and u_i's are nonzero real numbers that make the Julia invariant θ_0 minimal.

Let A_i be the zero of the quadratic $(x - \alpha_i)^2$ and B_i the point in the upper half space representing the quadratic $(x - \beta_i)(x - \bar{\beta}_i)$. Then $A_1, \ldots, A_r$ are the points on the real line with their x-coordinate equal to $\alpha_1, \ldots, \alpha_r$ and $B_1, \ldots, B_s$

are points in the upper half plane with coordinates $(\mathrm{Re}(\beta_i), \mathrm{Im}(\beta_i))$. Attach to the $A_1, \dots, A_r$, $B_1, \dots, B_s$ respectively the weights $t_1^2, \dots, t_r^2, 2u_1^2, \dots, 2u_s^2$. Construct the smallest convex polygon which contains on its boundary or its interior the A_i's together with their respective masses. This polygon obtained this way by the roots of the forms will be called the polygon associated to the form f. Then the following is true.

Lemma 8. *The zero map $\zeta : V_{n,\mathbb{R}} \to \mathcal{H}_2$, as in Eq. (9), maps the binary form $f \in V_{n,\mathbb{R}}$ to the centroid of the polygon $A_1, \dots, A_r, B_1, \dots, B_s$ weighted respectively by $t_1^2, \dots, t_r^2, 2u_1^2, \dots, 2u_s^2$.*

Proof. Let $f \in V_{n,\mathbb{R}}$ be a binary form and $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s$ be its roots, then the quadratic associated to f is given as $Q(x, 1) = Ax^2 - 2Bx + C$ where

$$
\begin{aligned}
A &= \sum_{i=1}^r t_i^2 + \sum_{i=1}^s 2u_i^2, \\
B &= \sum_{i=1}^r t_i^2 \alpha_i + \sum_{i=1}^s 2u_i^2 (\beta_i + \bar{\beta}_i), \\
C &= \sum_{i=1}^r t_i^2 \alpha_i^2 + \sum_{i=1}^s 2u_i^2 \beta_i \bar{\beta}_i
\end{aligned}
\tag{35}
$$

are as computed in Eq. (23). By Eq. (12) the root in the upper half plane of this quadratic is

$$\xi(f) = -\frac{B}{2A} + \frac{\sqrt{|\mathfrak{D}_f|}}{2A} \cdot i.$$

Given as a point $z = (x, y) \in \mathcal{H}_2$ we have that

$$x = \frac{\sum_{i=1}^r t_i^2 \alpha_i + \sum_{i=1}^s 2u_i^2 (\beta_i + \bar{\beta}_i)}{\sum_{i=1}^r t_i^2 + \sum_{i=1}^s 2u_i^2}$$

and

$$||z||^2 = \frac{\sum_{i=1}^r t_i^2 \alpha_i^2 + \sum_{i=1}^s 2u_i^2 \beta_i \cdot \bar{\beta}_i}{\sum_{i=1}^r t_i^2 + \sum_{i=1}^s 2u_i^2}.$$

On the other side the centroid C_P of the convex polygon is

$$C_P = \frac{\sum_{i=1}^r t_i^2 A_i + \sum_{i=1}^s 2u_i^2 B_i}{\sum_{i=1}^r t_i^2 + \sum_{i=1}^s 2u_i^2}.$$

It is easy to prove that the real coordinate and the distance from the origin of the centroid C_P agrees respectively with x, and $||z||$ as computed above. This completes the proof. □

The following problems are interesting to consider.

Problem 2. *Let P_n be the polygon associated to a degree n binary form as explained above. Let $A_1, \dots, A_r, B_1, \dots, B_s$, such that $r + 2s = n$ be the vertices of the polygon with masses $w_1, \dots, w_n$ respectively. Find $w_1, \dots, w_n$ such that the quantity $\sum_{1=i<j=n} w_i w_j (\alpha_i - \alpha_j)^2$ is minimal.*

Problem 3. *Let $f \in V_{n,\mathbb{R}}$ be a binary form, $\alpha_1, \dots, \alpha_r$ its real roots and $\beta_1, \dots, \beta_s$ its complex roots. Construct the convex polygon which contains the roots in its boundary or its interior. Compute the centroid of this convex polygon and move it to the fundamental domain, using the generators of the modular group S, and*

T. How do the symmetric polynomials of the new form compare to the ones of the given form f.

Reduction of binary forms can be done from a purely geometric approach. In [23], Julia proved that the point $\zeta(f)$ is tied to the roots of the form f with relations of non-euclidean geometry that are preserved when the modular group acts on them. Hence, this point is a covariant of the roots of the form for all modular transformations.

This geometric approach is helpful for forms which have special properties that make it easy to determine this point, as explained in [23].

Binary forms with complex coefficients. Let $f(x, 1)$ be a degree n binary form with complex coefficients and roots $\alpha_1, \dots, \alpha_n$. Associate to it the quadratic Hermitian

$$H(x, 1) = \sum_{i=1}^{n} t_i^2 \cdot ||x - \alpha_i||^2 \tag{36}$$

where the t_i's are nonzero real number such that make the Julia's invariant θ_0 minimal. We want to solve this optimization problem geometrically.

Let P_i be the zero in the upper half space of the quadratic $||x - \alpha_i||^2$, i.e. $P_1, \dots, P_n$ are points in the upper half space given as follows

$$P_i = (z_i, r_i) = (\alpha_i, 0).$$

Attach to $P_1, \dots, P_n$ the masses $t_1^2, \dots, t_n^2$ respectively. Construct the smallest convex polyhedron which contains in the boundary or its interior the P_i's together with their respective masses. This polyhedron obtained this way by the roots of the forms will be called the polyhedron associated to the form f. Then the following is true.

Lemma 9. *The point $\zeta(f)$, which is the zero in the upper half space $\mathcal{H}_3$ of the quadratic given in Eq. (36) is the centroid of this polyhedron.*

Proof. Let $f \in V_{n,\mathbb{C}}$ be a binary form and $\alpha_1, \dots, \alpha_n$ be its roots, then the binary quadratic Hermitian form associated to it is given as

$$H(x, 1) = A||x||^2 - Bx - \bar{B}\bar{x} + C$$

where

$$A = \sum_{i=1}^{n} t_i^2, \qquad B = \sum_{i=1}^{n} t_i^2 \alpha_i, \qquad \bar{B} = \sum_{i=1}^{n} t_i^2 \bar{\alpha}_i, \qquad C = \sum_{i=1}^{n} t_i^2 ||\alpha_i||^2$$

are as computed in Eq. (23). By Eq. (12) the root in the upper half space of this binary quadratic Hermitian form is

$$\xi(f) = -\frac{B}{A} + \frac{\sqrt{\mathfrak{D}_f}}{A} \cdot j.$$

or equivalently the point $P = (z, r) \in \mathcal{H}_3$ such that the projection in the complex plane is

$$\frac{\sum_{i=1}^{n} t_i^2 \alpha_i}{\sum_{i=1}^{n} t_i^2}$$

and the distance from the origin is

$$\frac{\sum_{i=1}^{n} t_i^2 ||\alpha_i||^2}{\sum_{i=1}^{n} t_i^2}.$$

On the other side the centroid C_P of the convex polygon is

$$C_P = \frac{\sum_{i=1}^n t_i^2 P_i}{\sum_{i=1}^n t_i^2}.$$

It is easy to prove that the distance of this point C_P from the origin and its projection to the complex plane agree with the ones of $\xi(f)$. This completes the proof. □

The problem of finding the Julia quadratic in this way can be formulated as follows.

Problem 4. *Let P_n be the polyhedron associated to a degree n binary form as above. Let $A_1, \dots, A_n$ be the vertices of the polyhedron with masses $w_1, \dots, w_n$ respectively. Find $w_1, \dots, w_n$ such that the centroid of the polyhedron is invariant under $\mathrm{SL}_2(\mathbb{C})$ action and makes the quantity $\sum_{1=i<j=n} w_i w_j \alpha_i \alpha_j$ (where α_i are complex numbers that we get by the projection of A_i's in the complex plane) minimal.*

In an analogues way with the previous section another interesting problem to consider here is the following.

Problem 5. *Let K be a number field and $\mathcal{O}_K$ its ring of integers. Let $f \in V_{n,K}$ be a binary form, $\alpha_1, \dots, \alpha_n$ its roots. Construct the convex polyhedron which contains in the boundary or its interior the roots of the given form. Compute the centroid of this convex polyhedron and move it to the fundamental domain, when such exists. How do the symmetric polynomials of the new form compare to the ones of the given form f.*

While there is a huge amount of literature on optimization problems of this type, we are not aware of any specific results that apply to this situation.

6.2. **Totally real forms.** Let $f \in V_{n,\mathbb{R}}$ such that f has signature $(n, 0)$. Such forms are called **totally real forms**. Let f be a generic form in $V_{n,\mathbb{R}}$ given by

$$f(x, y) = a_n x^n + a_{n-1} x^{n-1} y + \cdots + a_1 x y^{n-1} + a_0 y^n$$

where $a_0, \dots, a_n$ are transcendentals. Identify $a_0, \dots, a_n$ respectively with $1, \dots, n+1$. Then the symmetric group S_{n+1} acts on $\mathbb{R}[a_0, \dots a_n][x, y]$ by permuting $a_0, \dots, a_n$. For any permutation $\tau \in S_{n+1}$ and $f \in \mathbb{R}[a_0, \dots a_n][x, y]$ we denote by $\tau(f) = f^\tau$. Then

$$f^\tau(x, y) = \tau(a_n)\, x^n + \tau(a_{n-1})\, x^{n-1} y + \cdots + \tau(a_1)\, x y^{n-1} + \tau(a_0)\, y^n.$$

Define $G(x, y)$ as follows

$$G(x, y) = \frac{x \cdot f_x(-f_y(x, y), f_x(x, y)) + y \cdot f_y(-f_y(x, y), f_x(x, y))}{n\, f(x, y)}. \tag{37}$$

In [34] Stoll and Cremona was proved that $G(x, y)$ is a degree $d = (n-1)(n-2)$ homogenous polynomial in $\mathbb{R}[a_0, \dots a_n][x, y]$ and $\mathcal{J}_f(x, y) \mid G(x, y)$; see [34] for details. Therefore, this polynomial can be used to reduce totally real binary forms.

Note that, for $\sigma \in S_{n+1}$ we have an involution

$$\sigma = \begin{cases} (1, n+1)(2, n) \cdots \left(\frac{n}{2}, \frac{n}{2} + 2 \right), & \text{if } n \text{ is even} \\ (1, n+1)(2, n) \cdots \left(\frac{n+1}{2}, \frac{n+3}{2} \right), & \text{if } n \text{ is odd.} \end{cases}$$

The polynomial $G(x, y)$ satisfies the following.

Theorem 10. *Let $f \in V_{n,\mathbb{R}}$ with distinct roots, sig $(f) = (n, 0)$, and G_f as above. Then*

i) $G(x, y)$ is a covariant of f of degree $(n-1)$ and order $(n-1)(n-2)$.

ii) $G(x, y)$ has a unique quadratic factor over $\mathbb{R}$, which is $\mathcal{J}_f$.

iii) $G^\sigma(x, y) = (-1)^{n-1} G(x, y)$. Moreover, if $G_f = \sum_{i=1}^d g_i x^i y^{d-i}$, then

$$g_i^\sigma = (-1)^{n-1} g_{d-i},$$

for all $i = 0, ..., d$.

Proof. The fact that $G(x, y)$ is a polynomial is a direct consequence of the Euler's theorem on homogenous functions and it is shown in [34].

Let $F(x, y) = (f, xy)^1$ be the 1-transvection. It is a covariant of order n and degree 1. From Euler's theorem of homogenous functions we have that $F(x, y) = xf_x + yf_y = nf(x, y)$.

Let us denote by $A = -f_y(x, y)$ and $B = f_x(x, y)$. Both are covariants of f of order $(n-1)$ and degree 1. Then $f_x(A, B) = \sum_{i=0}^n i\, a_i\, A^{i-1} B^{n-i}$ has degree n as a covariant and similarly for $f_y(A, B)$. Therefore,

$$\frac{xf_x(A, B) + yf_y(A, B)}{nf(x, y)}$$

is a covariant of degree $(n-1)$. Obviously, it has order $d = (n-1)(n-2)$. This completes the proof of part i). Part ii) is a restatement of the result proved in [34].

To prove part iii), it is enough to show that $g_i^\sigma = (-1)^{n-1} g_{d-i}$ for all $i = 0, ..., d$. Let $\tau = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. If we show that $G^\tau(x, y) = G(x, y)$, then this immediately implies that $g_i^\sigma = (-1)^{n-1} g_{d-i}$ for all $i = 0, ..., d$.

For any binary form $F(x, y)$, we have that $F^\tau(x, y) = F(-y, -x) = (-1)^n F(x, y)$. Such a rule also applies to f_x and f_y in Eq. (37). It is now elementary to verify that $G(-y, -x) = G(x, y)$. This completes the proof. $\square$

Next, we compute this covariant G_f for some small degree binary forms.

Example 1 (Cubics). *When $n = 3$, $G(x, y)$ is the Hessian of the binary cubic. It is given by the formula*

$$G_f = (3a_3a_1 - a_2^2)x^2 + (9a_3a_0 - a_2a_1)xy + (3a_2a_0 - a_1^2)y^2$$

when $f = \sum_{i=0}^3 a_i x^i y^{3-i}$. This formula was known to Stoll and Cremona in [34].

The permutation σ is $\sigma = (a_0, a_3)(a_1, a_2)$. Notice that $G^\sigma = G$. Moreover, $\Delta_G = -3 \cdot \Delta_f$.

Totally real quartics. Let $f \in V_{4,\mathbb{R}}$ with distinct real roots and

$$f(x, y) = \sum_{i=0}^4 a_i x^i y^{4-i}.$$

Then G_f is a degree 6 homogenous polynomial given as follows

$$G_f = \sum_{i=0}^6 g_i x^i y^{6-i},$$

where

$$g_6 = -\left(a_3^3 + 4\, a_2 a_3 a_4 - 8\, a_1 a_4^2\right)$$

$$
\begin{aligned}
g_5 &= 2\left(a_2 a_3^2 + 16\, a_0 a_4^2 + 2\, a_4 a_1 a_3 - 4\, a_4 {a_2}^2\right)\\
g_4 &= 5\ \left(a_1 a_3^2 - 4\, a_4 a_1 a_2 + 8\, a_4 a_0 a_3\right)\\
g_3 &= 20\ \left(a_0 a_3^2 -\ a_1^2 a_4\right)\\
g_2 &= -5\ \left(a_3 a_1^2 - 4\, a_0 a_3 a_2 + 8\, a_0 a_4 a_1\right)\\
g_1 &= -2\left(a_2 a_1^2 + 16\, a_4 a_0^2 + 2\, a_0 a_3 a_1 - 4\, a_0 {a_2}^2\right)\\
g_0 &= a_1^3 + 4\, a_2 a_0 a_1 - 8\, a_0^2 a_3
\end{aligned}
$$

In this case $\sigma = (15)(24)$, which in terms of the coefficients $a_0, \dots, a_4$ becomes $\sigma := (a_0, a_4)(a_1, a_3)$. Then it is easy to check that

$$
\sigma(g_6) = -g_0, \quad \sigma(g_5) = -g_1, \quad \sigma(g_4) = -g_2, \quad \sigma(g_3) = -g_3.
$$

The discriminant of G in terms of $a_0, \dots, a_4$ is given by

$$
\begin{aligned}
\Delta_G = &-2^{28}\ \big(a_1^2 {a_2}^2 a_3^2 - 4\, a_1^2 {a_2}^3 a_4 - 4\, a_1^3 a_3^3 + 18\, a_1^3 a_2 a_4 a_3 - 27\, a_1^4 a_4^2\\
&- 4\, a_0 {a_2}^3 a_3^2 + 16\, a_0 {a_2}^4 a_4 + 18\, a_0 a_1 a_3^3 a_2 - 80\, a_0 a_1 a_3 a_4 {a_2}^2 - 6\, a_0 a_1^2 a_3^2 a_4\\
&+ 144\, a_1^2 a_2 a_0 a_4^2 - 27\, a_0^2 a_3^4 + 144\, a_0^2 a_4 a_2 a_3^2 - 128\, a_0^2 {a_2}^2 a_4^2\\
&-192\, a_0^2 a_1 a_3 a_4^2 + 256\, a_0^3 {a_4}^3\big)^5
\end{aligned}
$$

It is easily verified that

$$
\Delta_G = -2^{28} \cdot \Delta_f^5 = -4^{14} \cdot \Delta_f^5.
$$

Remark 4. *We expect that in general $\Delta_G = r \cdot \Delta_f^{n+1}$, for some constant r. This fact has not been noticed before by previous authors.*

Totally real quintics. Let $f \in V_{5,\mathbb{R}}$ be a binary quintic $f = \sum_{i=0}^5 a_i x^i y^{5-i}$. Its signature is one of the following $\operatorname{sig}(f) = \{(5,0), (3,1), (1,2)\}$.

Assume that $\operatorname{sig}(f) = (r,s) = (5,0)$. In the notation of the previous section, we have $Q_f = T_5$. The discriminant of T_5 in terms of the roots α_i of the form is given by the formula

$$
\begin{aligned}
\Delta(T_5) = -4t_1^2 \cdots t_5^2 \bigg(&\frac{(\alpha_1-\alpha_2)^2}{t_3^2 t_4^2 t_5^2} + \frac{(\alpha_1-\alpha_3)^2}{t_2^2 t_4^2 t_5^2} + \frac{(\alpha_1-\alpha_4)^2}{t_3^2 t_2^2 t_5^2} + \frac{(\alpha_1-\alpha_5)^2}{t_2^2 t_3^2 t_4^2}\\
&+ \frac{(\alpha_2-\alpha_3)^2}{t_1^2 t_4^2 t_5^2} + \frac{(\alpha_2-\alpha_4)^2}{t_1^2 t_3^2 t_5^2} + \frac{(\alpha_2-\alpha_5)^2}{t_1^2 t_3^2 t_4^2} + \frac{(\alpha_3-\alpha_4)^2}{t_1^2 t_2^2 t_5^2}\\
&+ \frac{(\alpha_3-\alpha_5)^2}{t_1^2 t_2^2 t_4^2} + \frac{(\alpha_4-\alpha_5)^2}{t_1^2 t_2^2 t_3^2}\bigg).
\end{aligned}
$$

In this case $\sigma = (1,6)(2,5)(3,4)$ which correspond to $\sigma = (a_0, a_5)(a_1, a_4)(a_2, a_3)$. Then computing $G(x,y)$ as in Eq. (37) we have

$$
G(x,y) = c_{12} x^{12} + c_{11} x^{11} y + \cdots c_1 x y^{11} + c_0 y^{12} \tag{38}
$$

where the coefficients are given as follows:

$$
\begin{aligned}
c_{12} &= 125\, a_1 {a_5}^3 - 50\, a_2 a_4 {a_5}^2 + 15\, a_3 a_4^2 a_5 - 3\, a_4^4\\
c_{11} &= 625\, a_0 {a_5}^3 + 175\, a_1 a_4 {a_5}^2 - 100\, a_2 a_3 {a_5}^2 - 55\, a_2 a_4^2 a_5 + 60\, a_3^2 a_4 a_5 - 9\, a_3 a_4^3\\
c_{10} &= 1375\, a_0 a_4 {a_5}^2 - 25\, a_1 a_3 {a_5}^2 + 65\, a_1 a_4^2 a_5 - 150\, {a_2}^2 {a_5}^2 - 30\, a_2 a_3 a_4 a_5\\
&\quad - 41\, a_2 a_4^3 + 60\, a_3^3 a_5 + 3\, a_3^2 a_4^2
\end{aligned}
$$

$$
\begin{aligned}
c_9 &= 875\, a_0a_3{a_5}^2 + 1025\, a_0a_4^2a_5 - 425\, a_1a_2{a_5}^2 + 30\, a_1a_3a_4a_5 - 33\, a_1a_4^3 \\
&\quad - 130\, {a_2}^2a_4a_5 + 160\, a_2a_3^2a_5 - 86\, a_2a_3a_4^2 + 24\, a_3^3a_4 \\
c_8 &= 125\, a_0a_2{a_5}^2 + 1500\, a_0a_3a_4a_5 + 215\, a_0a_4^3 - 425\, a_1^2{a_5}^2 - 450\, a_1a_2a_4a_5 \\
&\quad + 175\, a_1a_3^2a_5 - 128\, a_1a_3a_4^2 + 175\, {a_2}^2a_3a_5 - 97\, {a_2}^2a_4^2 + 8\, a_2a_3^2a_4 + 12\, a_3^4 \\
c_7 &= -750\, a_0a_1{a_5}^2 + 500\, a_0a_2a_4a_5 + 775\, a_0a_3^2a_5 + 430\, a_0a_3a_4^2 - 530\, a_1^2a_4a_5 \\
&\quad + 310\, a_1a_2a_3a_5 - 322\, a_1a_2a_4^2 - 61\, a_1a_3^2a_4 + 105\, {a_2}^3a_5 - 33\, {a_2}^2a_3a_4 + 32\, a_2a_3^3 \\
c_6 &= -625\, a_0^2{a_5}^2 - 800\, a_0a_1a_4a_5 + 1200\, a_0a_2a_3a_5 + 30\, a_0a_2a_4^2 + 365\, a_0a_3^2a_4 \\
&\quad + 30\, a_1^2a_3a_5 - 303\, a_1^2a_4^2 + 365\, a_1{a_2}^2a_5 - 268\, a_1a_2a_3a_4 + a_1a_3^3 + {a_2}^3a_4 + 37\, {a_2}^2a_3^2 \\
c_5 &= -750\, a_0^2a_4a_5 + 500\, a_0a_1a_3a_5 - 530\, a_0a_1a_4^2 + 775\, a_0{a_2}^2a_5 + 310\, a_0a_2a_3a_4 \\
&\quad + 105\, a_0{a_3}^3 + 430\, a_1^2a_2a_5 - 322\, a_1^2a_3a_4 - 61\, a_1{a_2}^2a_4 - 33\, a_1a_2a_3^2 + 32\, {a_2}^3a_3 \\
c_4 &= 125\, a_0^2a_3a_5 - 425\, a_0^2a_4^2 + 1500\, a_0a_1a_2a_5 - 450\, a_0a_1a_3a_4 + 175\, a_0{a_2}^2a_4 \\
&\quad + 175\, a_0a_2a_3^2 + 215\, a_1^3a_5 - 128\, a_1^2a_2a_4 - 97\, a_1^2a_3^2 + 8\, a_1{a_2}^2a_3 + 12\, {a_2}^4 \\
c_3 &= 875\, {a_0}^2a_2a_5 - 425\, a_0^2a_3a_4 + 1025\, a_0a_1^2a_5 + 30\, a_0a_1a_2a_4 - 130\, a_0a_1a_3^2 \\
&\quad + 160\, a_0{a_2}^2a_3 - 33\, a_1^3a_4 - 86\, a_1^2a_2a_3 + 24\, a_1{a_2}^3 \\
c_2 &= 1375\, a_0^2a_1a_5 - 25\, a_0^2a_2a_4 - 150\, a_0^2a_3^2 + 65\, a_0a_1^2a_4 - 30\, a_0a_1a_2a_3 + 60\, a_0{a_2}^3 \\
&\quad - 41\, a_1^3a_3 + 3\, a_1^2{a_2}^2 \\
c_1 &= 625\, a_0^3a_5 + 175\, a_0^2a_1a_4 - 100\, a_0^2a_2a_3 - 55\, a_0a_1^2a_3 + 60\, a_0a_1{a_2}^2 - 9\, a_1^3a_2 \\
c_0 &= 125\, a_0^3a_4 - 50\, a_0^2a_1a_3 + 15\, a_0a_1^2a_2 - 3\, a_1^4
\end{aligned}
$$

The following is an immediate consequence of Thm. 10.

Corollary 4. *Let $f \in V_{5,\mathbb{R}}$ with signature $(5,0)$. Then $G^\sigma = G$ the above coefficients give a computational confirmation that $G^\sigma = G$ and $c_i^\sigma = c_{5-i}$ for all $i = 1, \dots, 5$.*

Next we will study binary forms where all the roots are complex and will see the similarity of such forms with totally real forms.

6.3. **Totally complex forms.** Let $f(x,y) \in \mathbb{R}$ be a degree $n = 2s$ binary form with signature $(0,s)$. We will call such forms **totally complex forms**. Then $f(x,y)$ can be factored as follows

$$
f(x,1) = \prod_{i=1}^{s} (x - \alpha_i)(x - \bar{\alpha}_i) = \prod_{i=1}^{s} (x^2 + A_i x + B_i).
$$

and assume $\alpha_i = a_i + I\, b_i$, for $i = 1, \dots s$. Associate to it the quadratic

$$
S(x,y) = 2 \sum_{i=1}^{s} u_i^2 (x^2 + A_i xy + B_i y^2).
$$

The discriminant of $S(x,1)$ is computed in Eq. (20) and is

$$
\Delta_S = -16 \left(\sum_{i<j} u_i^2 u_j^2 \left[(a_i - a_j)^2 + (b_i^2 + b_j^2) \right] + \sum_{j=1}^{s} u_j^4 b_j^2 \right)
$$

In analogy with the theory explained in Section 5.1 let $h = \Delta_S$ and $g = u_1^4 \cdots u_s^4$. We have to solve the following system

$$\begin{cases} \nabla_h = \lambda \nabla_g \\ u_1^4 \cdots u_s^4 = 1 \end{cases}$$

for $\lambda \neq 0$. For all $i = 1, \dots, s$ the partial derivative of Δ_S with respect to u_i is

$$-16 \left(2 \sum_{i<j} u_i u_j^2 \left[(a_i - a_j)^2 + (b_i^2 + b_j^2) \right] + 4 \sum_{j=1}^{s} u_j^3 b_j^2 \right)$$

and the above system Eq. (6.3) is

$$\begin{cases} -16 \left(2 \sum_{i<j} u_1 u_j^2 \left[(a_1 - a_j)^2 + (b_1^2 + b_j^2) \right] + 4 \sum_{j=1}^{s} u_j^3 b_j^2 \right) = 4\lambda u_1^3 \cdots u_s^4 \\ \qquad \vdots \\ -16 \left(2 \sum_{i<j} u_s u_j^2 \left[(a_s - a_j)^2 + (b_s^2 + b_j^2) \right] + 4 \sum_{j=1}^{s} u_j^3 b_j^2 \right) = 4\lambda u_1^4 \cdots u_s^3 \\ u_1^4 \cdots u_s^4 = 1 \end{cases} \tag{39}$$

To find the point in the upper half plane that is used for reduction we need to find the unique solution of the above system. Next, we compute the Julia quadratic of totally complex binary quartics and sextics.

Totally complex quartics. Let f be a binary quartic with signature $(0, 2)$ and factored as follows $f(x, y) = \sum_{i=1}^{2} (x^2 + a_i xy + b_i y^2)$. Associate to f the quadratic Q_f, where

$$Q_f(x, 1) = 2u_1^2(x^2 + a_1 x + b_1) + 2u_2^2(x^2 + a_2 x + b_2)$$

To find u_1, and u_2 we set up the system as in Eq. (39) and solve for the u_i's. The discriminant of the quadratic which is

$$\Delta_Q = {a_1}^2{u_1}^2 + 2\,a_1a_2u_1u_2 + {a_2}^2{u_2}^2 - 4\,b_1{u_1}^2 - 4\,b_1u_1u_2 - 4\,b_2u_1u_2 - 4\,b_2{u_2}^2.$$

Next, compute the partial derivatives of Δ_Q with respect to u_1, u_2, and then set up the system. This is done in Maple but we do not display the system here. The system is given in terms of u_i's, a_i's, b_i's and λ, the Lagrange multiplier. Solving for λ we get

$$\lambda = -2\,\frac{({a_1}^2 - a_1a_2 - 2\,b_1 + 2\,b_2)x^2 + 2x(\,a_1b_1 - 2\,a_2b_1 + a_1b_2) + {a_1}^2b_2 - a_1a_2b_1 + 2\,{b_1}^2 - 2\,b_1b_2}{x^2 + xa_1 + b_1}$$

Substitute λ as computed in the system (39) and add to this system the equation $Q(x, 1) = 0$. Using this approach we can compute the point $\xi(f)$ in the upper half plane corresponding to the Julia quadratic. Eliminating u_1, and u_2 we get a degree 4 polynomial

$$G_f = \sum_{i=0}^{4} c_i x^i y^{4-i}, \tag{40}$$

with coefficients as follows

$$\begin{aligned}
c_4 &= -2\,{a_1}^2 + 2\,{a_2}^2 + 8\,b_1 - 8\,b_2\\
c_3 &= -4\,{a_1}^2 a_2 + 4\,a_1 {a_2}^2 - 16\,a_1 b_2 + 16\,a_2 b_1\\
c_2 &= -12\,{a_1}^2 b_2 + 12\,{a_2}^2 b_1\\
c_1 &= -4\,{a_1}^2 a_2 b_2 + 4\,a_1 {a_2}^2 b_1 - 16\,a_1 b_1 b_2 + 16\,a_2 b_1 b_2\\
c_0 &= -2\,{a_1}^2 {b_2}^2 + 2\,{a_2}^2 {b_1}^2 - 8\,{b_1}^2 b_2 + 8\,b_1 {b_2}^2.
\end{aligned}$$

If we let $b_1 = b_2 = 1$ then $G_f(x, y)$ is a palindromic polynomial, i.e.

$$\begin{aligned}
c_4 = c_0 &= -2\,{a_1}^2 + 2\,{a_2}^2\\
c_3 = c_1 &= -4\,{a_1}^2 a_2 + 4\,a_1 {a_2}^2 - 16a_1 + 16a_2\\
c_2 &= -12\,{a_1}^2 + 12\,{a_2}^2.
\end{aligned}$$

This degree 4 polynomial has a unique quadratic factor which is the Julia's quadratic and will be used to reduce the given form.

Totally complex sextics. Let f be a binary sextic with signature $(0, 3)$ and factored as follows

$$f(x, y) = (x^2 + a_1 xy + b_1 y^2)(x^2 + a_2 xy + b_2 y^2)(x^2 + a_3 xy + b_3 y^2).$$

Associate to f the quadratic

$$Q(x, 1) = 2u_1^2(x^2 + a_1 x + b_1) + 2u_2^2(x^2 + a_2 x + b_2) + 2u_3^2(x^2 + a_3 x + b_3)$$

where the u_i's are real numbers that make θ_f minimal. To find u_1, u_2 and u_3 that satisfy this condition we need to set up the system in eq (39) and solve for the u_i's. Compute first the discriminant of the quadratic which is as follows

$$\begin{aligned}
\Delta_Q =& 4\,{a_1}^2{u_1}^4 + 8\,a_1 a_2 {u_1}^2{u_2}^2 + 8\,a_1 a_3 {u_1}^2{u_3}^2 + 4\,{a_2}^2{u_2}^4 + 8\,a_2 a_3 {u_2}^2{u_3}^2\\
&+ 4\,{a_3}^2{u_3}^4 - 16\,b_1{u_1}^4 - 16\,b_1{u_1}^2{u_2}^2 - 16\,b_1{u_1}^2{u_3}^2 - 16\,b_2{u_1}^2{u_2}^2\\
&- 16\,b_2{u_2}^4 - 16\,b_2{u_2}^2{u_3}^2 - 16\,b_3{u_1}^2{u_3}^2 - 16\,b_3{u_2}^2{u_3}^2 - 16\,b_3{u_3}^4
\end{aligned}$$

Next, compute the partial derivatives of Δ_Q with respect to u_1, u_2, and u_3 and then set up the system. This is done in Maple but we do not display the system here because is to big. The system is given in terms of u_i's, a_i's, b_i's and λ, the Lagrange multiplier. Solving for λ we get

$$\lambda = \frac{4(a_3{u_1}^2a_1 + a_3{u_2}^2a_2 + {u_3}^2{a_3}^2 - 2\,{u_1}^2b_1 - 2\,{u_2}^2b_2 - 2\,b_3{u_1}^2 - 2\,b_3{u_2}^2 - 4\,{u_3}^2b_3)}{{u_3}^2{u_1}^4{u_2}^4}$$

Substitute λ as computed in the system (39) and add to this system the equation $Q(x, 1) = 0$. Using this approach we can compute the point $\xi(f)$ in the upper half plane corresponding to the Julia quadratic. Computationally it is too difficult to eliminate all three u_1, u_2, and u_3 at the same time, so first we eliminate u_1, and u_2 and then at a second step eliminate u_3. Eliminating all three of them we get a degree 8 polynomial

$$G_f = \sum_{i=0}^{8} c_i x^i y^{8-i}, \tag{41}$$

with coefficients given in [6]. This degree 8 polynomial has a unique quadratic factor which is the Julia quadratic and will be used to reduce the given form.

As a special case, consider the case when we let $b_1 = b_2 = b_3 = 1$. The binary form f is given as follows

$$f(x,y) = \prod_{i=1}^{3} (x^2 + a_i xy + y^2).$$

The function $G_f(x,y)$ associated to this binary form has coefficients as follows

$$\begin{aligned}
c_8 &= -c_0 = 3\,(a_2 - a_3)(a_1 - a_3)(a_1 - a_2)(a_1a_2 + a_1a_3 + a_2a_3)\\
c_7 &= -c_1 = 3\,(a_2 - a_3)(a_1 - a_3)(a_1 - a_2)(3\,a_1a_2a_3 + 8\,a_1 + 8\,a_2 + 8\,a_3)\\
c_6 &= -c_2 = 6\,(a_2 - a_3)(a_1 - a_3)(a_1 - a_2)(5\,a_1a_2 + 5\,a_1a_3 + 5\,a_2a_3 + 24)\\
c_5 &= -c_3 = 9\,(a_2 - a_3)(a_1 - a_3)(a_1 - a_2)(a_1a_2a_3 + 8\,a_1 + 8\,a_2 + 8\,a_3)\\
c_4 &= 0.
\end{aligned}$$

Note that $G_f(x,y)$ is a palindromic polynomial.

Remark 5. *If $f = \prod_{i=1}^{n}(x^2 + a_i xy + y^2)$, then f is a palindromic form. In this case, the Julia quadratic is a factor of G_f, where G_f is also a palindromic form.*

6.4. **Julia's quadratic of binary forms of small degree.** We give examples when $n = 3, 4$ to illustrate the theory in Section 5 and to show explicitly how the u_i's can be determined using the system given in Eq. (28).

Binary cubic forms. Let $f \in V_{3,\mathbb{C}}$ and let denote its roots by $\alpha_1, \alpha_2, \alpha_3$, then

$$f(x,1) = a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3), \tag{42}$$

for some $a_0 \in \mathbb{C}$. We associate to f the following positive definite quadratic form

$$\begin{aligned}
Q_f(\alpha,1) =& t_1^2(x-\alpha_1)(\bar{x}-\bar{\alpha}_1) + t_2^2(x-\alpha_2)(\bar{x}-\bar{\alpha}_2) + t_3^2(x-\alpha_3)(\bar{x}-\bar{\alpha}_3)\\
=& \left(t_1^2 + t_2^2 + t_3^2\right) x\bar{x} - \left(t_1^2\bar{\alpha}_1 + t_2^2\bar{\alpha}_2 + t_3^2\bar{\alpha}_3\right) x\\
& - \left(t_1^2\alpha_1 + t_2^2\alpha_2 + t_3^2\alpha_3\right)\bar{x} + \left(t_1^2||\alpha_1||^2 + t_2^2||\alpha_2||^2 + t_3^2||\alpha_3||^2\right).
\end{aligned} \tag{43}$$

The discriminant of $Q_f(x,y)$ is given as follows

$$\Delta_Q = {t_1}^2{t_2}^2\,||\alpha_1 - \alpha_2|\,|^2 + t_1^2t_3^2||\alpha_1 - \alpha_3||^2 + t_2^2t_3^2||\alpha_2 - \alpha_3||^2.$$

For simplicity in the computations denote by $h = \Delta_Q$, then compute its gradient and replace $\alpha_{i,j} := ||\alpha_i - \alpha_j||^2$. We have

$$\nabla_h = \left\langle\, 2t_1t_2^2\alpha_{12} + 2t_1t_3^2\alpha_{13},\ 2t_2t_1^2\alpha_{12} + 2t_2t_3^2\alpha_{23},\ 2t_3t_1^2\alpha_{13} + 2t_3t_2^2\alpha_{23} \right\rangle.$$

As in Section 5.1 we take $g = t_1^2 \cdot t_2^2 \cdot t_3^2 = 1$, and its gradient is

$$\nabla_g = \left\langle\, 2t_1t_2^2t_3^2,\ 2t_2t_1^2t_3^2,\ 2t_3t_1^2t_2^2 \right\rangle.$$

Then the system in Eq. (5.1) is as follows

$$\left\{\begin{array}{c}
2t_1t_2^2\alpha_{12} + 2t_1t_3^2\alpha_{13} = 2\lambda t_1t_2^2\,t_3^2\\
2t_2t_1^2\alpha_{12} + 2t_2t_3^2\alpha_{23} = 2\lambda t_2t_1^2\,t_3^2\\
2t_3t_1^2\alpha_{13} + 2t_3t_2^2\alpha_{23} = 2\lambda t_3t_1^2\,t_2^2\\
t_1^2 \cdot t_2^2 \cdot t_3^2 = 1.
\end{array}\right.$$

Simplifying and substituting $t_1^2 \cdot t_2^2 \cdot t_3^2 = 1$ we have

$$\begin{cases} t_1^2(t_2^2\alpha_{12} + t_3^2\alpha_{13}) = \lambda \\ t_2^2(t_1^2\alpha_{12} + t_3^2\alpha_{23}) = \lambda \\ t_3^2(t_1^2\alpha_{13} + t_2^2\alpha_{23}) = \lambda \\ t_1^2 \cdot t_2^2 \cdot t_3^2 = 1. \end{cases}$$

Substitute $\lambda = \frac{2}{3} \cdot (u_1u_2\alpha_{1,2} + u_2u_3\alpha_{2,3} + u_1u_3\alpha_{1,3})$ as in Eq. (27), write everything in terms of u's and $\alpha_{i,j}$'s, and then combine like terms

$$\begin{cases} u_1u_2\alpha_{12} + u_1u_3\alpha_{13} = 2u_2u_3\alpha_{2,3} \\ u_1u_2\alpha_{12} + u_2u_3\alpha_{23} = 2u_1u_3\alpha_{1,3} \\ u_1u_3\alpha_{13} + u_2u_3\alpha_{23} = 2u_1u_2\alpha_{1,2} \\ u_1 \cdot u_2 \cdot u_3 = 1. \end{cases}$$

We further normalize by letting $\alpha_{1,2} \cdot \alpha_{1,3} \cdot \alpha_{2,3} = 1$. Solving the above system for u_1, u_2, u_3 we get

$$\begin{cases} u_1 = \frac{\alpha_{1,2}}{\alpha_{1,3}{u_3}^2} \\ u_2 = \frac{\alpha_{1,3}u_3}{\alpha_{1,2}} \\ \alpha_{1,3}u_3\left({\alpha_{1,2}}^2 - \alpha_{1,3}\alpha_{2,3}{u_3}^3\right) = 0. \end{cases}$$

Consider $\alpha_{1,3}u_3\left({\alpha_{1,2}}^2 - \alpha_{1,3}\alpha_{2,3}{u_3}^3\right) = 0$. Since, $u_i \neq 0$ we have

$$\left({\alpha_{1,2}}^2 - \alpha_{1,3}\alpha_{2,3}{u_3}^3\right) = 0.$$

Multiply both sides of the above with $\alpha_{1,2}$ and then making the substitution $\alpha_{1,2} \cdot \alpha_{1,3} \cdot \alpha_{2,3} = 1$ we get $u_3^3 = \alpha_{1,2}^3$. By the definition of the quadratic $Q(x,1)$ associated to $f(x,1)$, all u_i are non zero real numbers, then this equation $u_3^3 = \alpha_{1,2}^3$ has a unique real solution, namely $u_3 = \alpha_{1,2}$. Therefore, the unique solution to the above system is

$$u_1 = \alpha_{2,3},\ u_2 = \alpha_{1,3},\ u_3 = \alpha_{1,2}.$$

Substituting these values of u_1, u_2, u_3, which are the values that minimize θ_0, into Eq. (43) we get Julia's quadratic $\mathcal{J}_f$.

$$\mathcal{J}_f(x,1) = ||\alpha_2 - \alpha_3||^2(x-\alpha_1)^2 + ||\alpha_1 - \alpha_3||^2(x-\alpha_2)^2 + ||\alpha_1 - \alpha_2||^2(x-\alpha_3)^2.$$

Let p, q, r denote the coefficients of Julia quadratic, then they are respectively

$$p := 2\,{\alpha_1}^2 - 2\,\alpha_1\alpha_2 - 2\,\alpha_1\alpha_3 + 2\,{\alpha_2}^2 - 2\,\alpha_2\alpha_3 + 2\,{\alpha_3}^2$$

$$q := -2\,{\alpha_1}^2\alpha_2 - 2\,{\alpha_1}^2\alpha_3 - 2\,\alpha_1{\alpha_2}^2 + 12\,\alpha_1\alpha_2\alpha_3 - 2\,\alpha_1{\alpha_3}^2 - 2\,{\alpha_2}^2\alpha_3 - 2\,\alpha_2{\alpha_3}^2$$

$$r := 2\,{\alpha_1}^2{\alpha_2}^2 - 2\,{\alpha_1}^2\alpha_2\alpha_3 + 2\,{\alpha_1}^2{\alpha_3}^2 - 2\,\alpha_1{\alpha_2}^2\alpha_3 - 2\,\alpha_1\alpha_2\alpha_3^2 + 2\,{\alpha_2}^2{\alpha_3}^2$$

Now, let f be a generic cubic given as follows

$$f(x,1) = ax^3 + bx^2 + cx + d$$

where $a = a_0$, $b = \alpha_1 + \alpha_2 + \alpha_3$, $c = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$, and $d = \alpha_1\alpha_2\alpha_3$. Eliminating the roots we can express Julia's quadratic coefficient in terms of the coefficient of $f(X)$ as follows

$$p = b^2 - 3ac,\ q = bc - 9ad,\ r = c^2 - 3bd$$

up to a constant factor.

Notice that $\mathfrak{D}_f = -3 \cdot \Delta_f$, where Δ_f is the discriminant of the cubic. In this case the discriminant of the Julia quadratic is an $\mathrm{SL}_2(\mathbb{Z})$ invariant of the binary form. We summarize as follows:

Lemma 10. *Let f be a stable binary cubic with equation*

$$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3.$$

Then its Julia quadratic is given by

$$\mathcal{J}_f = (b^2 - 3ac)\,x^2 + (bc - 9ad)\,xy + (c^2 - 3bd)y^2 \tag{44}$$

and its discriminant is $\Delta(J_f) = -3\,\Delta_f$, where Δ_f is the discriminant of f. Moreover, its θ-invariant is

$$\theta_f = a_0^6\, 3^{\frac{3}{2}}\; |\Delta_f|^{\frac{1}{2}}.$$

As we will see in the next section the situation is more complicated for forms of higher degree.

Binary quartics. We illustrate the case of binary quartics. Let f be a binary quartic given as follows

$$f(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \tag{45}$$

and $\alpha_1, \dots, \alpha_4$ its roots when $y = 1$. To this binary quartic we associate a positive definite quadratic form as follows

$$Q(x,1) = \sum_{i=1}^{4} t_i^2 (x - \alpha_i)^2 \tag{46}$$

and its discriminant as computed in Eq. (24)

$$\Delta_Q = \sum_{1=i<j=4} u_i\, u_j\, \alpha_{i,j}$$

where as above $u_i = t_i^2$ and $\alpha_{i,j} = (\alpha_i - \alpha_j)^2$, for $i < j$. Compute the gradient of the discriminant of $Q(x,1)$ with respect to u_i's. Then the system in Eq. (25) in this case will be

$$\begin{cases} u_1(\alpha_{1,2}u_2 + \alpha_{1,3}u_3 + \alpha_{1,4}u_4) = (1/2)\,\sum_{1=i<j=4} u_i\, u_j\, \alpha_{i,j} \\ u_2(\alpha_{1,2}u_1 + \alpha_{2,3}u_3 + \alpha_{2,4}u_4) = (1/2)\,\sum_{1=i<j=4} u_i\, u_j\, \alpha_{i,j} \\ u_3(\alpha_{1,3}u_1 + \alpha_{2,3}u_2 + \alpha_{3,4}u_4) = (1/2)\,\sum_{1=i<j=4} u_i\, u_j\, \alpha_{i,j} \\ u_4(\alpha_{1,4}u_1 + \alpha_{2,4}u_2 + \alpha_{3,4}u_3) = (1/2)\,\sum_{1=i<j=4} u_i\, u_j\, \alpha_{i,j}. \\ u_1u_2u_3u_4 - 1 = 0. \end{cases}$$

Combining like terms and simplifying we have

$$\begin{cases} u_1u_2\alpha_{1,2} + u_1u_3\alpha_{1,3} + u_1u_4\alpha_{1,4} - u_2u_3\alpha_{2,3} - u_2u_4\alpha_{2,4} - u_3u_4\alpha_{3,4} = 0 \\ u_1u_2\alpha_{1,2} + u_2u_3\alpha_{2,3} + u_2u_4\alpha_{2,4} - u_1u_3\alpha_{1,3} - u_1u_4\alpha_{1,4} - u_3u_4\alpha_{3,4} = 0 \\ u_1u_3\alpha_{1,3} + u_2u_3\alpha_{2,3} + u_3u_4\alpha_{3,4} - u_1u_2\alpha_{1,2} - u_1u_4\alpha_{1,4} - u_2u_4\alpha_{2,4} = 0 \\ u_1u_4\alpha_{1,4} + u_2u_4\alpha_{2,4} + u_3u_4\alpha_{3,4} - u_1u_2\alpha_{1,2} - u_1u_3\alpha_{1,3} - u_2u_3\alpha_{2,3} = 0 \\ u_1u_2u_3u_4 - 1 = 0. \end{cases}$$

We want to solve the above system for $u_1, \dots, u_4$. Add up the first and second equation, then first and third, and lastly first and fourth to get

$$\begin{cases} u_1u_2\alpha_{1,2} = u_3u_4\alpha_{3,4} \\ u_1u_3\alpha_{1,3} = u_2u_4\alpha_{2,4} \\ u_1u_4\alpha_{1,4} = u_2u_3\alpha_{2,3}. \end{cases}$$

Then multiplying each side of the above system we get

$$u_1^3 = u_2u_3u_4 \cdot \frac{\alpha_{2,3}\alpha_{2,4}\alpha_{3,4}}{\alpha_{1,2}\alpha_{1,3}\alpha_{1,4}}.$$

Multiplying both sides with u_1 we get

$$u_1^4 = \frac{\alpha_{2,3}\alpha_{2,4}\alpha_{3,4}}{\alpha_{1,2}\alpha_{1,3}\alpha_{1,4}}.$$

The same way we can compute u_2^4, u_3^4, u_4^4. This proves the following.

Lemma 11. *The degree of the field extension* $[k(u_1, \dots, u_4) : k(\alpha_{i,j})] = 4$. *Moreover,*

$$\begin{aligned} u_1^4 &= \frac{\alpha_{2,3}\alpha_{2,4}\alpha_{3,4}}{\alpha_{1,2}\alpha_{1,3}\alpha_{1,4}}, \ u_2^4 = \frac{\alpha_{1,3}\alpha_{1,4}\alpha_{3,4}}{\alpha_{1,2}\alpha_{2,3}\alpha_{2,4}}, \\ u_3^4 &= \frac{\alpha_{1,2}\alpha_{1,4}\alpha_{2,4}}{\alpha_{1,3}\alpha_{2,3}\alpha_{3,4}}, \ u_4^4 = \frac{\alpha_{1,2}\alpha_{1,3}\alpha_{2,3}}{\alpha_{1,4}\alpha_{2,4}\alpha_{3,4}}. \end{aligned}$$

Hence, we have the following diagram of field extensions.

$$\begin{array}{c} k(\alpha_1, \dots, \alpha_4) \\ {\scriptstyle 6}\Big| \\ k(u_1, \dots, u_4) \\ {\scriptstyle 4}\Big| \\ k(\alpha_{i,j}) \end{array}$$

The proof of $[k(\alpha_1, \dots, \alpha_4) : k(u_1, \dots, u_4)] = 6$ is done computationally via Maple.

Remark 6. *For any given 4-tuple* $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *we have a unique positive real solution* (u_1, u_2, u_3, u_4). *Hence, as expected the coefficients of the Julia quadratic are uniquely defined.*

Therefore, the Julia quadratic of the given quartic is

$$\mathcal{J}_f = px^2 - 2qx + r \tag{47}$$

where p, q, and r are as follows

$$\begin{aligned} p \cdot d =& (\alpha_3 - \alpha_4)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4) + (\alpha_3 - \alpha_4)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4) + \\ & (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_4)(\alpha_1 - \alpha_4) + (\alpha_1 - \alpha_2)((\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3) \\ q \cdot d =& \alpha_1(\alpha_3 - \alpha_4)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4) + \alpha_2(\alpha_3 - \alpha_4)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4) + \\ & \alpha_3(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_4)(\alpha_1 - \alpha_4) + \alpha_4(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3) \\ r \cdot d =& \alpha_1^2(\alpha_3 - \alpha_4)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4) + \alpha_2^2(\alpha_3 - \alpha_4)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4) + \\ & \alpha_3^2(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_4)(\alpha_1 - \alpha_4) + \alpha_4^2(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3) \end{aligned}$$

where

$$d = \Big((\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_4)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_4) \Big)^{\frac{1}{2}}$$

Computing Julia's quadratic discriminant we have

$$\mathfrak{D}_f = \frac{-4\,a^3\,(\alpha_2 - \alpha_4)(\alpha_1 - \alpha_3)}{\Delta_f^{1/2}}.$$

In an analogues way with the cubics, we want to express Julia's quadratic coefficients in terms of the coefficients of the binary quartic form. Write down the symmetric polynomials, as well as the coefficients of Julia's quadratic and then eliminate the roots of the quartic $\alpha_1, \dots, \alpha_4$. In this case, the computations show that the coefficients of the Julia quadratic cannot be expressed nicely in terms of the coefficient of the binary quadratic, as in the case of binary cubics. The case of binary quintics and sextics is done in detail in [7]. The main results are given in the following theorems.

Theorem 11. *Let $f \in V_{5,\mathbb{R}}$. The quadratic Q_f associated to f is as follows.*

i) If sig $(f) = (5,0)$ then $Q_f = T_5$, where T_5 is the unique quadratic factor of Eq. (38)*.*

ii) If sig $(f) = (3,1)$ then $Q_f = T_3 + S_1$, where T_3 is the quadratic given in Eq. (44) *and S_1 as*

$$S_1 = 2u_1^2 \left(x^2 - 2\,\mathrm{Re}(\beta)x + ||\beta||^2\right)$$

for some $\beta \in \mathcal{H}_2$ such that $f(\beta) = 0$.

iii) If sig $(f) = (1,2)$ then $Q_f = T_1 + S_2$, where S_2 is the unique quadratic factor of Eq. (40) *and T_1 as follows*

$$T_1 = t_1^2 \left(x - \alpha\right)^2$$

for some $\alpha \in \mathbb{R}$ such that $f(\alpha) = 0$.

And for binary sextics we have the following.

Theorem 12. *Let $f \in V_{6,\mathbb{R}}$. The quadratic Q_f associated to f is as follows.*

i) If sig $(f) = (6,0)$ then $Q_f = T_6$, where T_6 is the unique quadratic factor of the equation given in [6, Appendix]*.*

ii) If sig $(f) = (4,1)$ then $Q_f = T_4 + S_1$, where T_4 is the quadratic given in Eq. (47) *and S_1 as*

$$S_1 = 2u_1^2 \left(x^2 - 2\,\mathrm{Re}(\beta)x + ||\beta||^2\right)$$

for some $\beta \in \mathcal{H}_2$ such that $f(\beta) = 0$.

iii) If sig $(f) = (2,2)$ then $Q_f = T_2 + S_2$, where T_2 is given as

$$T_2 = t_1^2 \left(x - \alpha_1\right)^2 + t_2^2 \left(x - \alpha_2\right)^2$$

and S_2 is the unique quadratic factor of Eq. (40)*.*

iv) If sig $(f) = (0,3)$ then $Q_f = S_3$, where S_3 is the unique quadratic factor of the equation given in [6, Appendix]*.*

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