ALBANIAN JOURNAL OF MATHEMATICS
Volume 12, Number 1, Pages 24–31
ISSN: 1930-1235; (**2018**)

A NEGATIVE ANSWER TO A QUESTION OF ASCHBACHER

ROBERT A. WILSON

School of Mathematical Sciences, Queen Mary University of London,
Mile End Road, London E1 4NS, U.K.

Dedicated to the memory of Kay Magaard

Abstract. We give infinitely many examples to show that, even for simple groups G, it is possible for the lattice of overgroups of a subgroup H to be the Boolean lattice of rank 2, in such a way that the two maximal overgroups of H are conjugate in G. This answers negatively a question posed by Aschbacher.

MSC 2010: Primary: 20E32, 20E34; Secondary: 20D05, 20D06
Keywords: Simple groups, maximal subgroups

1. The question

In a recent survey article on the subgroup structure of finite groups [1], in the context of discussing open problems on the possible structures of subgroup lattices of finite groups, Aschbacher poses the following specific question. Let G be a finite group, H a subgroup of G, and suppose that H is contained in exactly two maximal subgroups M_1 and M_2 of G, and that H is maximal in both M_1 and M_2. Does it follow that M_1 and M_2 are not conjugate in G? This is Question 8.1 in [1]. For G a general group, he asserts there is a counterexample, not given in [1], so he restricts this question to the case G almost simple, that is $S \leq G \leq \mathrm{Aut}(S)$ for some simple group S. This is Question 8.2 in [1].

2. The answer

In fact, the answer is no, even for simple groups G. Two or three examples can be read off from the Atlas of Finite Groups [4], if one knows where to look. For convenience, let us call a group G an *A-group* if the question has an affirmative answer for G, and a *non-A-group* otherwise. The smallest example of a non-A-group seems to be the simple Mathieu group M_{12} of order 95040.

E-mail address: `r.a.wilson@qmul.ac.uk`.

Theorem 1. *Let $G = \mathrm{M}_{12}$, and $H \cong A_5$ acting transitively on the 12 points permuted by M_{12}. Then H lies in exactly two other subgroups of G, both lying in the single conjugacy class of maximal subgroups $L_2(11)$.*

Proof. The maximal subgroups of G are well-known, and are listed in the Atlas of Finite Groups [4, p. 33]. From this list it follows that the only maximal subgroups of G that contain H are conjugates of the transitive subgroup $M \cong L_2(11)$. The maximal subgroups of $\mathrm{Aut}(G) \cong \mathrm{M}_{12}{:}2$ are determined in [9], where it is shown in particular that the normalizers in $\mathrm{Aut}(G)$ of H and M are $\mathrm{Aut}(H) \cong S_5$ and $\mathrm{Aut}(M) \cong PGL_2(11)$ respectively. Since $PGL_2(11)$ does not contain S_5, it follows that there are precisely two conjugates of M that contain H, and that these conjugates are interchanged by elements of $\mathrm{Aut}(H) \setminus H$. □

Of course, one example answers the specific question, but does not address the context in which the question was asked. One needs to consider rather how many examples there are, or whether the phenomenon just exhibited is relatively common or rare. The problem considered in [9] was to what extent the maximal subgroups of an automorphism group $\mathrm{Aut}(S)$ of a simple group S can be deduced from those of the simple group itself. The biggest obstruction to such a reduction turns out to be the existence of what were called *type* 2 *novelties*, that is maximal subgroups of $\mathrm{Aut}(S)$ whose intersection with S, say H, lies in exactly one conjugacy class of maximal subgroup of S.

In fact, type 2 novelties are a good source of examples of non-A-groups, although the maximality of H in M_i is an extra condition that needs to be checked separately. Indeed, the results of [9] can be used to deduce the existence of one more example, that is in the sporadic simple group of Held. The relevant subgroup information, obtained in [3, 9], is summarised in [4, p. 104].

Theorem 2. *Let $G = \mathrm{He}$ and $H \cong (A_5 \times A_5).2.2$. Then H is contained in just two other subgroups of G, both lying in the single class of maximal subgroups isomorphic to $S_4(4){:}2$.*

Proof. It is shown in [3] (see also [9]) that there is a unique class of $A_5 \times A_5$ in the Held group, and that the normalizer of any $A_5 \times A_5$ in G is a group H of shape $(A_5 \times A_5).2^2$, in which there is no normal A_5. It follows that H lies inside a maximal subgroup $M \cong S_4(4){:}2$. Now in $\mathrm{Aut}(G)$ the normalizers of H and M are $S_5 \wr 2$ and $S_4(4){:}4$ respectively. But $S_4(4){:}4$ does not contain $S_5 \wr 2$, so the elements of $\mathrm{Aut}(H) \setminus H$ interchange two G-conjugates of M that contain H. □

The above examples constitute the extent of general knowledge at the time of the publication of the Atlas.

3. Doubly-deleted doubly-transitive permutation representations

Both the examples given so far occur in sporadic groups G. There is also at least one example in which H is sporadic, but G is a classical simple group.

Theorem 3. *Let $G = \Omega_{10}^-(2)$ and $H \cong \mathrm{M}_{12}$. Then H is contained in exactly two other subgroups of G, both lying in the single conjugacy class of subgroups isomorphic to A_{12}.*

Proof. In this case there is a crucial error in [4, p. 147] and one needs to use the corrected list of maximal subgroups of G from [6] or [2]. Note in particular that

$\mathrm{Aut}(G) \cong \Omega_{10}^{-}(2){:}2$ contains maximal subgroups S_{12} and $\mathrm{Aut}(\mathrm{M}_{12}) \cong \mathrm{M}_{12}{:}2$. Since S_{12} does not contain $\mathrm{M}_{12}{:}2$, we have essentially the same situation as in the two previous examples. The only maximal subgroups of G that contain H are conjugates of $M \cong A_{12}$, and there are exactly two such conjugates, swapped by elements of $\mathrm{Aut}(H) \setminus H$. □

Analysing this example, it is clear that an important property of M_{12} that is being used here is that it has two distinct 2-transitive representations on 12 points, swapped by the outer automorphism. The smallest simple group with such a property is $L_3(2)$, which has two distinct 2-transitive representations on 7 points. In characteristic 7, therefore, there is a doubly-deleted permutation representation, giving rise to an embedding in $\Omega_5(7)$.

Theorem 4. *Let $G = \Omega_5(7)$ and $H \cong L_3(2) \cong L_2(7)$ be a subgroup of G, acting irreducibly on the 5-dimensional module. Then H is contained in exactly two other subgroups of G, both isomorphic to A_7, and lying in the same G-conjugacy class.*

Proof. Reading off the information about irreducible subgroups of $\Omega_5(7)$ and $SO_5(7)$ from [2, Table 8.23], we see that $G = \Omega_5(7)$ has a single class of irreducible subgroups $H = L_3(2)$, and these subgroups are contained in maximal subgroups $M = A_7$. Correspondingly, in $SO_5(7)$ there are maximal subgroups $L_3(2){:}2$ and S_7. The outer automorphism of $L_3(2)$ therefore swaps two (G-conjugate) copies of A_7 containing $L_3(2)$. □

More generally, for all $n \geq 3$ and all prime powers q, the simple group $L_n(q)$ has two inequivalent 2-transitive permutation representations on $d := (q^n - 1)/(q - 1)$ points. Not all of these give rise to examples of non-A-groups, however. The case $L_3(3) < A_{13} < \Omega_{11}(13)$ can be analysed using the classification of maximal subgroups of orthogonal groups in 11 dimensions in [2], where we find that $\Omega_{11}(13)$ contains two classes of $L_3(3){:}2$, so that $L_3(3)$ embeds in both A_{13} and $L_3(3){:}2$. Similarly, the cases $L_4(2) < A_{15} < \Omega_{13}(\ell)$ for $\ell = 3, 5$ are described in [7]. There is one class of A_8, and two classes of S_{15}, in $\Omega_{13}(3)$, so A_8 is not second maximal in this case. There is one class of A_{15}, and two classes of S_8, in $\Omega_{13}(5)$, so A_8 is contained in three maximal subgroups in this case.

4. Infinite series of examples

If p is a prime bigger than 7, then there is an embedding $L_3(2) < A_7 < \Omega_6^{\varepsilon}(p)$, where $\varepsilon = +$ just when p is a quadratic residue modulo 7. For simplicity, restrict to the case $\varepsilon = +$. We read off the following properties from [2, Table 8.9]. The number of classes of A_7 is at least 2, and is exactly 2 when $p \equiv 3 \bmod 4$. The same is true for $L_3(2)$. In this case, the centre of $\Omega_6^+(p)$ is trivial, and the outer automorphism group has order 4, consisting of a diagonal automorphism δ, a graph automorphism γ, and their product $\delta\gamma$. Now A_7 is normalized by γ in all cases, while $L_3(2)$ is normalized by γ provided $p \equiv \pm 1 \bmod 8$, and by $\delta\gamma$ otherwise. Thus we must restrict to the case $p \equiv 7 \bmod 8$, and $p \equiv 1, 2, 4 \bmod 7$, that is $p \equiv 15, 23, 39 \bmod 56$. In these cases, the group $\Omega_6^+(p).\langle\gamma\rangle = SO_6^+(p)$ contains two classes of S_7, and two classes of $L_3(2){:}2$. The automorphism δ swaps the two classes of S_7, and swaps the two classes of $L_3(2){:}2$.

Theorem 5. *Let p be a prime, and suppose that $p \equiv 15, 23, 39 \bmod 56$. Let G be the simple group $\Omega_6^+(p) \cong PSL_4(p)$. Then G contains subgroups $L_3(2) < A_7$,* *both normalized by the tranpose-inverse automorphism of $L_4(p)$, to $L_3(2){:}2$ and S_7 respectively. In particular, every such $L_3(2)$ lies in exactly two copies of A_7, and these two copies of A_7 are G-conjugate.*

Proof. Consider a pair of subgroups $L_3(2) < A_7$ of G, and adjoin $\alpha\gamma$, where α is an inner automorphism of $\Omega_6^+(p)$, to extend A_7 to S_7. This swaps the two classes of $L_3(2)$ in A_7. But we can also adjoin $\beta\gamma$, where β is another inner automorphism, to normalize $L_3(2)$ to $L_3(2){:}2$. Hence there is an inner automorphism of the form $\alpha\gamma\beta\gamma$, that conjugates an $L_3(2)$ of one class in A_7, to an $L_3(2)$ of the other class. The same argument with the roles of $L_3(2)$ and A_7 reversed shows that the two copies of A_7 in which $L_3(2)$ lies are conjugate in $\Omega_6^+(p)$. $\square$

As a consequence, we have an infinite series of groups $PSL_4(p)$, for p any prime with $p \equiv 15, 23, 39 \bmod 56$, for which Aschbacher's question has a negative answer. There is a similar infinite series of groups $PSU_4(p)$, for $p \equiv 1 \bmod 8$ and $p \equiv 3, 5, 6 \bmod 7$, that is $p \equiv 17, 33, 41 \bmod 56$. This can be read off in a similar way from [2, Table 8.11].

Theorem 6. *Let p be a prime, and suppose that $p \equiv 17, 33, 41 \bmod 56$. Let G be the simple groups $\Omega_6^-(p) \cong PSU_4(p)$. Then G contains subgroups $L_3(2) < A_7$, both normalized by the field automorphism of $U_4(p)$, to $L_3(2){:}2$ and S_7 respectively. In particular, every such $L_3(2)$ lies in exactly two copies of A_7, and these two copies of A_7 are conjugate in G.*

These last two results are essentially contained in [2, Proposition 4.8.4], where the fact that type 2 novelties arise in these cases is proved. The maximality of H in M_i is a triviality. The authors of [2] remark that type 2 novelties also arise for other values of p, but the conditions on p cannot be expressed as simple congruence conditions. There is an analogous embedding $L_2(11) < A_{11}$, which one might think gives similar series of examples in $\Omega_{10}^{\varepsilon}(p)$ for certain p. However, $L_2(11)$ is not maximal in A_{11}, so this fails.

5. More special examples

As we have just seen, the embedding $L_3(2) < A_7$ behaves differently in characteristic 7 (the *special* case) from other characteristics (the *generic* case). More generally, the embedding $L_n(q) < A_d$, where $d = (q^n - 1)/(q - 1)$, behaves differently in the special case (characteristic dividing d), compared to the generic case (characteristic prime to d).

The special case is easiest to analyse when d is itself prime. In this case, n is necessarily prime, but q need not be prime. This includes all Mersenne primes except 3, and others such as $(3^3 - 1)/(3 - 1) = 13$ and $(5^3 - 1)/(5 - 1) = 31$, for example. We then have embeddings $L_n(q) < A_d < \Omega_{d-2}(d)$. The Singer cycles in $L_n(q)$ are represented as d-cycles in A_d, and as regular unipotent elements in $\Omega_{d-2}(d)$. Now there is a unique class of regular unipotent elements in $SO_m(d)$ for all odd m, and these elements have order d provided $m \leq d$. The class splits into two classes in $\Omega_m(d)$, and these classes are rational if $m \equiv \pm 1 \bmod 8$, and irrational otherwise.

Since d is prime, the d-cycles in S_d split into two irrational classes in A_d (by Sylow's Theorem). The d-cycles are conjugate in A_d to their inverses just when $d \equiv 1 \pmod 4$. Since the regular unipotent elements have unipotent centralizer, it follows that they are conjugate in $\Omega_{d-2}(d)$ to their inverses if and only if either $d \equiv 1 \bmod 4$ or $d-2 \equiv \pm 1 \bmod 8$, that is $d \equiv 1, 3, 5 \bmod 8$. Now the Singer cycles in $L_n(q)$ are inverted by the transpose-inverse automorphism, and we want this automorphism to be realised by an element of $SO_{d-2}(d) \setminus \Omega_{d-2}(d)$. This happens if and only if $d \equiv 7 \bmod 8$.

Theorem 7. *If q is a prime power, and $d := (q^n-1)/(q-1)$ is prime, with $d \equiv 7 \bmod 8$, let $H = P\Gamma L_n(q)$, $M = A_d$ and $G = \Omega_{d-2}(d)$. Then $H < M < G$, and H and M are unique up to conjugacy in G. Hence H and M extend to $H.2$ and $M.2$ in $G.2$, and H is contained in exactly two G-conjugates of M.*

The condition $d \equiv 7 \bmod 8$ is satisfied by all Mersenne primes (the case $q = 2$), except 3, but not by all primes of the form $(q^n-1)/(q-1)$. The condition can be re-written as a condition on the values of q and n modulo 8.

Lemma 1. *If $d = (q^n-1)/(q-1)$, then the condition $d \cong 7 \bmod 8$ is equivalent to the condition that, either*

- $q = 2$ *and* $n > 2$, *or*
- $q \equiv 1 \bmod 8$ *and* $n \equiv 7 \bmod 8$, *or*
- $q \equiv 5 \bmod 8$ *and* $n \equiv 3 \bmod 8$.

Only finitely many primes d of the form $(q^n-1)/(q-1)$ are known, but it is conjectured that there are infinitely many, including infinitely many Mersenne primes 2^n-1. Currently just 50 Mersenne primes are known, giving rise to examples with H isomorphic to $L_3(2)$, $L_5(2)$, $L_7(2)$, $L_{13}(2)$, ..., $L_{77232917}(2)$. Less effort has been expended on finding primes for larger values of q, but examples for $q = 5$ and $n \equiv 3 \bmod 8$ occur when $n = 3$, 11, 3407, 16519, 201359 and 1888279 (see A004061 in the On-line Encyclopedia of Integer Sequences [8]). I could find no examples with $q = 9$ or $q = 13$, but using GAP [5], one can easily find the examples $n = 7$, 47 and 71 for $q = 17$ and $n \equiv 7 \bmod 8$.

One can also search for examples by fixing n rather than q. For $n = 3$, examples with $q \equiv 5 \bmod 8$ and d prime include $q = 5, 101, 173, 293, 677, 701, 773$. A search with $n = 7$ turns up the examples $q = 17, 73, 89, 353, 1297, 1409, 1489, 1609$, 1753, 2609, $2753, 3673, 4049, 4409$, etc., and similarly for $n = 11$, we can take $q = 53, 229, 389, 709, 1213, 2029, 5581, 5669, 5813, 5861, 7229$. For $n = 19$, there are examples for $q = 181, 277, 389, 509, 797, 1693, 1709$, etc. For $n = 23$, $q = 113$, 257, 857, 1801; for $n = 31$, $q = 241$, and so on.

In particular, examples of negative answers to Aschbacher's question arise in the cases of $L_3(5)$, $L_3(101)$, $L_{11}(5)$, $L_7(17)$, and $L_7(73)$. An extremely large example arises from the embedding of $L_{77232917}(2)$ in A_d and $\Omega_{d-2}(d)$, where $d = 2^{77232917}-1$ is the largest currently known Mersenne prime.

6. More generic examples

As we have seen, for all $n \geq 3$ and for all q, the simple groups $L_n(q)$ have two inequivalent permutation representations on $d := (q^n-1)/(q-1)$ points, and hence we obtain two inequivalent embeddings in A_d. In the generic case, when ℓ is a prime not dividing d, the alternating group A_d embeds irreducibly into $\Omega_{d-1}(\ell)$. However, the conditions on n, q, ℓ for this to give rise to a negative answer to Aschbacher's question, are subtle and complicated, as we already saw for the smallest case, $n = 3, q = 2$.

The next smallest case is $n = 3, q = 3$. To analyse this case, that is, the embedding $L_3(3) < A_{13} < P\Omega_{12}^{\pm}(p)$, we may use the information on maximal subgroups of $\Omega_{12}^{\pm}(p)$ provided in [2, Tables 8.83 and 8.85]. It follows from these tables that there are no examples here.

The next smallest case is $n = 4, q = 2$, and the embedding of $L_4(2) \cong A_8$ into A_{15} and thence into orthogonal groups in dimensions 13 and 14. The maximal subgroups of these orthogonal groups have been determined by Anna Schroeder, in her St Andrews PhD thesis [7]. In particular, the embeddings into $\Omega_{13}(3)$ and $\Omega_{13}(5)$ do not give examples.

In the dimension 14 case, however, it seems that there is a small but crucial error at exactly the point that interests us here: maximal subgroups S_8 are eliminated from the lists of maximal subgroups of $P\Omega_{14}^{\pm}(p)$ by the assertion, contained in the proof of [7, Propn. 6.4.17(iv)], that $S_8 \le S_{15}$, which is manifestly false for this embedding. Indeed, Propositions 6.4.4 and 6.4.5 in [7] give the true picture, and show that S_8 is indeed a maximal subgroup of $SO_{14}^{\varepsilon}(p)$ for suitable congruences of ε and p. In the cases when the outer automorphism group of $\Omega_{14}^{\varepsilon}(p)$ is just 2^2, the calculations are quite straightforward. These are the cases when $\varepsilon p \equiv 3 \bmod 4$.

Theorem 8. *Let $p \equiv 19, 23, 31, 47 \bmod 60$, and let $G = \Omega_{14}^{+}(p)$. Let $H \cong A_8$ be a subgroup of G acting irreducibly in the 14-dimensional representation. Then H is contained in exactly two maximal subgroups of G, both isomorphic to A_{15}, and conjugate to each other in G.*

Proof. Indeed, it is shown in [7] that for $p \equiv 19, 23, 31, 47 \bmod 60$, there are two conjugacy classes of subgroups S_{15}, maximal in $SO_{14}^{+}(p)$, and swapped by the diagonal automorphism δ. Moreover, it is shown that the intersection of S_{15} with $\Omega_{14}^{+}(p)$ is A_{15}. Now the same argument applies to the group S_8, acting irreducibly in the 14-dimensional representation. Since for this embedding, S_8 does not lie in S_{15}, it follows that S_8 is maximal in $SO_{14}^{+}(p)$ in these cases. □

Exactly the same argument applies to the cases $p \equiv 13, 29, 37, 41 \bmod 60$ in $\Omega_{14}^{-}(p)$. It is possible that analogous examples also exist when $\varepsilon p \equiv 1 \bmod 4$, but in this case the outer automorphism group is D_8, and there are four classes each of S_8 and S_{15}, so the situation is more complicated.

7. Unbounded rank

From what we have done so far, if there are infinitely many Mersenne primes, then there are examples of non-A-groups of arbitrarily large Lie rank. However, in this section we shall show that this condition can be removed, by considering the generic rather than the special case.

Note first that, for n even, the representations of $L_n(2)$ of dimension $d-1$, where $d = 2^n - 1$, extend to emebeddings of $L_n(2){:}2$ in $SO_d(p)$ for all p, while for n odd this happens only when the field of order p contains square roots of 2, that is, when $p \equiv \pm 1 \bmod 8$. Hence, for example, the embeddings $L_5(2) < A_{31} < \Omega_{30}^{\varepsilon}(p)$ provide examples of non-A-groups whenever all of the following conditions are satisfied:

- $\varepsilon p \equiv 3 \bmod 4$,
- $p \equiv \pm 1 \bmod 8$, and
- $p \equiv 1, 2, 4, 8, 16 \bmod 31$.

That is to say, for $\varepsilon = +$ we require $p \equiv 39, 47, 63, 95, 159 \bmod 248$, while for $\varepsilon = -$ we require $p \equiv 1, 33, 97, 225, 233 \bmod 248$.

For the purpose of demonstrating that examples of non-A-groups exist of arbitrarily large rank, and of arbitrarily large characteristic within a given rank, it is sufficient to consider any infinite subset of such primes. For simplicity, we restrict to the case when $\varepsilon = -$, and further to the case when $p \equiv 1 \bmod 4(d-1)$. In this case, the embedding of $L_n(2)$ into A_d and thence into $\Omega^-_{d-1}(p)$ gives an example of a negative answer to Aschbacher's question. Of course, there are many other examples.

Theorem 9. *Let p be a prime, and $\varepsilon = \pm$, such that $\varepsilon p \equiv 3 \bmod 4$. Let $n \geq 3$, and suppose that p is a square modulo $d := 2^n - 1$. If n is odd, suppose also that $p \equiv \pm 1 \bmod 8$. Let $G = \Omega^\varepsilon_{d-1}(p)$, and $H \cong L_n(2)$ a subgroup of G. Then H is contained in exactly two maximal subgroups of G, which are isomorphic to A_d and conjugate to each other.*

Proof. The above conditions ensure that G has outer automorphism group of order 4, and that both $L_n(2){:}2$ and S_d embed in $SO^\varepsilon_{d-1}(p)$ but not in $\Omega^\varepsilon_{d-1}(p)$. Hence we have the same configuration as in all the other examples above. □

We have now shown that there is no bound on the Lie rank of non-A-groups. In these examples, there are two conjugacy classes of $L_n(2)$ in $\Omega^\varepsilon_{d-1}(p)$, and two conjugacy classes of A_d, interchanged by the diagonal automorphism. If instead $\varepsilon p \equiv 1 \bmod 4$, then there are four classes of each, and the outer automorphism group of $\Omega^\varepsilon_{d-1}(p)$ is D_8. In [2], two of the reflections in D_8 are described as graph automorphisms γ, and the other two as $\delta\gamma$, but unfortunately the two conjugates of γ are not distinguished from each other. For any particular choice of γ, two of the four classes of A_d extend to S_d, and the other two classes are interchanged.

8. Other classical groups

So far, all our examples with G a classical group have occurred when G is in fact orthogonal. There is no bound on the characteristic, and there is no bound on the rank. All three families of orthogonal groups (plus type, minus type, and odd dimension) occur. It would be interesting to know if the other classical groups, linear, unitary or symplectic, can occur.

Of course, the isomorphisms $L_4(p) \cong \Omega^+_6(p)$ and $U_4(p) \cong \Omega^-(p)$ imply the existence of examples in linear and unitary groups, but do examples exist in linear and unitary groups of larger dimension? So far, I have not found any examples. The large outer automorphism groups in these cases make the analysis very delicate. There are potential examples of the form $L_3(4) < U_4(3) < L_6(p)$, but there are three classes of each of $L_3(4)$ and $U_4(3)$, and the embeddings between them are not given explicitly in [2]. Hence one needs extra detailed information to resolve these cases. It seems likely, however, that this configuration does not give any examples.

In the case of symplectic groups, over fields of odd prime order, the outer automorphism group has order 2, which is the ideal situation for us. If one looks through the tables of maximal subgroups of symplectic groups in dimensions up to 12 given in [2], one finds, besides the case $L_3(2) < A_7 < S_4(7)$ already discussed, just one series of potential examples, given by the embeddings $A_5 < L_2(p) < S_6(p)$ for p a prime, $p \equiv \pm 11, \pm 19 \bmod 40$. However, in this case the embedding of $2\dot{A}_5$ in $Sp_6(p)$ also goes via the tensor product $Sp_2(p) \circ GO_3(p)$, so this A_5 lies in more than two maximal subgroups of $Sp_6(p)$.

On the other hand, Anna Schroeder's PhD thesis [7] contains the lists of maximal subgroups of $S_{14}(q)$ and their automorphism groups. There one finds two more potential infinite series of examples, given by the embeddings $J_2 < S_6(p) < S_{14}(p)$ and $L_2(13) < S_6(p) < S_{14}(p)$ for suitable primes p. The relevant congruences are $p \equiv \pm 11, \pm 19 \bmod 40$ for J_2, and $p \equiv \pm 3, \pm 27, \pm 29, \pm 35, \pm 43, \pm 51 \bmod 104$ for $L_2(13)$. It is straightforward to check, in the same way as before, that these do indeed give examples of negative answers to Aschbacher's question.

Theorem 10. *Let $p \equiv \pm 11, \pm 19 \bmod 40$, and let $G = S_{14}(p)$. Let $H \cong \mathrm{J}_2$ be a subgroup of G. Then H is contained in exactly two maximal subgroups of G, both isomorphic to $S_6(p)$, and conjugate to each other in G.*

Theorem 11. *Let $p \equiv \pm 3, \pm 27, \pm 29, \pm 35, \pm 43, \pm 51 \bmod 104$, and let $G = S_{14}(p)$. Let $H \cong L_2(13)$ be a subgroup of G contained in $M \cong S_6(p)$. Then H is contained in exactly two maximal subgroups of G, both conjugate to M.*

9. Further remarks

Far-reaching as the above examples are, they have little, if any, impact on Aschbacher's programme. This is because they all occur in sporadic or classical groups, whereas Aschbacher is only proposing to use this approach for exceptional groups of Lie type. Our examples therefore merely show that his question is still too broad, and that the question needs to be restricted to a smaller class of groups than the class of almost simple groups.

The maximal subgroups are known completely for five of the ten families of exceptional groups of Lie type, and, of the remaining five, E_8 seems least likely to be a source of examples, since it admits neither diagonal nor graph automorphisms. Similarly, F_4 admits no diagonal automorphisms, and admits a graph automorphism only in characteristic 2. Probably the most promising places to look for examples of non-A-groups are in E_6 with a graph automorphism, and in E_7 with a diagonal automorphism. On the other hand, it is entirely conceivable that every finite exceptional group of Lie type is an A-group.

References

[1] M. Aschbacher, The subgroup structure of finite groups, *Finite simple groups: thirty years of the Atlas and beyond*, Contemp. Math. **694**, 111–121, AMS, 2017.

[2] J. N. Bray, D. F. Holt and C. M. Roney-Dougal, *The maximal subgroups of the low-dimensional classical groups*, LMS Lecture Notes **407**, Cambridge Univ. Press, 2013.

[3] G. Butler, The maximal subgroups of the sporadic simple group of Held, *J. Algebra* **69**, 67–81, 1981.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An Atlas of Finite Groups*, Oxford Univ. Press, 1985.

[5] The GAP group, GAP – Groups, Algorithms, and Programming, Version 4.8.10; 2018. (https://www.gap-system.org)

[6] Ch. Jansen, K. Lux, R. A. Parker and R. A. Wilson, *An Atlas of Brauer Characters*, LMS Monographs **11**, Oxford Univ. Press, 1995.

[7] A. K. Schröder, The maximal subgroups of the classical groups in dimension 13, 14 and 15, PhD Thesis, University of St Andrews, 2015.

[8] N. J. A. Sloane, The On-line Encycopedia of Integer Sequences, https://oeis.org

[9] R. A. Wilson, Maximal subgroups of automorphism groups of simple groups, *J. London Math. Soc.* **32**, 460–466, 1985.

[10] Robert Wilson, Peter Walsh, Jonathan Tripp, Ibrahim Suleiman, Richard Parker, Simon Norton, Simon Nickerson, Stephen Linton, John Bray, Richard Barraclough and Rachel Abbott, *Atlas of Group Representations*, `http://brauer.maths.qmul.ac.uk/Atlas/v3/`, 2005–.